PART 2. GLOBAL RANDOM SEARCH

CHAPTER 3. MAIN CONCEPTS AND APPROACHES OF GLOBAL RANDOM SEARCH

The present chapter contains three sections. Section 3.1 describes and studies the simplest global random search algorithms, outlines the ways of constructing more efficient algorithms, presents a general scheme of global random search algorithms and discusses the connection between local optimization and global random search. Section 3.2 proves some general results on convergence. Section 3.3 is devoted to Markovian algorithms, that are thoroughly theoretically investigated in literature.

3.1 Construction of global random search algorithms:
Basic approaches

This section can be referred to as an introduction to the methodology of global random search. It describes a few simple global random search algorithms and ways of increasing their efficiency. The simplest algorithm is considered first.

3.1.1 Uniform random sampling

According to the general concept of global optimization, any global optimization algorithm has to search in all the feasible region $X$ in some way or another. The simplest of these ways is a uniform sampling in $X$ that can be accomplished in both deterministic (described in Section 2.2.1) and stochastic fashion. The simplest stochastic (global random search) method consists in choosing the points at which $f$ is evaluated randomly, independently and uniformly in $X$.

Algorithm 3.1.1. (Uniform random search in $X$).

1. Set $k=1$, $f_0*=\infty$.
2. Obtain a point $x_k$ by sampling from the uniform distribution on $X$.
3. Evaluate $f(x_k)$ and set

$$f_k^* = \min \left\{ f_{k-1}^*, f(x_k) \right\}. \quad (3.1.1)$$

4. If $k=N$, then terminate the algorithm; choose the point $x_k^*$ with $f(x_k^*)=f_k^*$ as an approximation for $x^*=\arg \min f$. If $k<N$, then return to Step 2 (substituting $k+1$ for $k$).

Algorithm 3.1.1 has some different names, viz., crude search, pure random search, random bombardment, Monte Carlo method, etc. It utilizes the simplest stopping rule
which terminates the algorithm after a given number $N$ of evaluations of $f$. Chapter 4 describes and studies a mathematical statistical apparatus that can be used as a basis of various stopping rules of Algorithm 3.1.1 and many others according to which an algorithm terminates by attaining a given accuracy.

While using Algorithm 3.1.1 in practice, it is usually profitable to descend locally from one or several points with the lowest function values obtained (this is true for almost all global random search algorithms, as will be discussed later in Section 3.1.5).

The simplicity of Algorithm 3.1.1 makes possible the direct investigation of its theoretical properties considered below.

Let $x^* = \arg \min f$ be a global minimizer of $f$. For the set $B(\varepsilon) = B(x^*, \varepsilon)$ we have

$$\mu_n \{ B(\varepsilon) \} \leq \mu_n \{ x \in \mathbb{R}^n : \|x\| \leq \varepsilon \} = \pi^{n/2} \varepsilon^n / \Gamma(n/2 + 1)$$

(3.1.2)

where $\mu_n$ is the Lebesgue measure, and (3.1.2) becomes an equality if $\{x \in \mathbb{R}^n : \|x - x^*\| \leq \varepsilon\} \subset X$, i.e. if the distance from $x^*$ to the boundary of $X$ is not less than $\varepsilon$. Using (3.1.2) we obtain for all $\varepsilon > 0$, $k = 1, 2, \ldots$

$$\Pr \left\{ \min_{1 \leq i \leq k} \|x_i - x^*\| \leq \varepsilon \right\} = \Pr \{ x \in B(\varepsilon) \} = \mu_n \{ B(\varepsilon) \} / \mu_n (X) \leq$$

$$\leq \pi^{n/2} \varepsilon^n / \left[ \mu_n (X) \Gamma(n/2 + 1) \right],$$

(3.1.3)

$$\Pr \left\{ \min_{1 \leq i \leq k} \|x_i - x^*\| \leq \varepsilon \right\} = 1 - \left( 1 - \mu_n \{ B(\varepsilon) \} / \mu_n (X) \right)^k \leq$$

$$\leq 1 - \left[ 1 - \pi^{n/2} \varepsilon^n / (\mu_n (X) \Gamma(n/2 + 1)) \right]^k \to 1, \ k \to \infty.$$  

(3.1.4)

The relation obtained shows that the sequence

$$\min_{1 \leq i \leq k} \|x_i - x^*\|$$

converges in probability to zero for $k \to \infty$. Moreover, there is an estimator of the convergence rate in (3.1.4). The estimator of the expected number of steps before hitting into the set $B(\varepsilon)$ is easily obtained from (3.1.3)

$$\mathbb{E} \tau_{B(\varepsilon)} = \frac{\mu_n (X)}{\mu_n \{ B(\varepsilon) \}} \geq \mu_n (X) \pi^{-n/2} \varepsilon^{-n} \Gamma(n/2 + 1)$$

(3.1.5)

where $\tau_A$ is the moment of first hit of the search sequence $x_1, x_2, \ldots$ into a set $A \subset X$. These formulas estimate the rate of convergence of Algorithm 3.1.1 with respect to values