ON A THREE-DIMENSIONAL
CONVECTIVE STEFAN PROBLEM
FOR A NON-NEWTONIAN FLUID

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Abstract: The coupling of diffusion and convection phenomena in a material under a change of phase is presented. The Stefan problem is adopt with convection only in the liquid phase. The existence result for weak solutions is proved.

Keywords: Non-Newtonian fluids, convective Stefan problem.

1. INTRODUCTION

In this work we consider the coupling of diffusion and convection phenomena in a material under a change of phase. As a model we adopt the Stefan problem, with convection only in the liquid phase. The main mathematical difficulty lies in the interesting and delicate question of defining the liquid zone and the corresponding formulation of the motion equations. The liquid zone can be defined as the set \( \{ \theta > 0 \} \), where \( \theta \) is the renormalized temperature. A natural requirement is the continuity of the temperature (or at least the lower semicontinuity) that leads to an open set \( \{ \theta > 0 \} \) where the equations for the velocity \( \mathbf{v} \) can be suitably formulated in a weak sense. To deal with this main issue we consider a non-Newtonian fluid of dilatant type for which \( \nabla \mathbf{v} \in L^q \) with \( q > N = 3 \). This restriction is sufficient to assure enough integrability of the convective term in the Stefan problem to obtain a continuous solution.

Other conduction–convection problems, similar to this one from the point of view of the mathematical analysis, have been considered previously. We mention in particular the works [CDK1], where the Stefan problem is coupled with the Navier–Stokes equations in a stationary setting, the extension to the evolutionary problem in
[CDK2], that was successful only for (the Stokes system and) two spatial variables, and the partial regularity obtained in [DO1] and [DO2] in the three-dimensional case, respectively for the Stokes and Navier–Stokes equations. Related contributions are given in [RI,2] and [RU], where the stationary problem is studied in a very general setting. In [RU] the Stefan problem for the $p$-Laplacian operator ($1 < p < \infty$) is coupled with a variational inequality modeling a non-Newtonian flow with $v \in W^{1,q}$, and a continuous temperature is obtained provided $pq > N$, $N$ being the spatial dimension, thus showing the existence of a weak solution.

For the problem considered here we obtain an existence result for weak solutions via an approximation and penalization procedure and the consequent passage to the limit using appropriate a priori estimates. We combine monotonicity methods with a local compactness argument needed to show the convergence of the nonlinear convective term in the flow equations. The continuity of the temperature is obtained using the techniques developed by DiBenedetto (see [DI,2]). The proof lies on energy and logarithmic estimates, that are possible due to the integrability of the velocity that follows from our essential assumption that $q > N = 3$.

2. THE MATHEMATICAL FORMULATION
AND MAIN RESULT

We consider a material occupying a bounded regular domain $\Omega \subset \mathbb{R}^3$ and coexisting in two phases, a solid phase, corresponding to a region $S$, and a liquid phase, corresponding to a region $L$. The two regions are separated by a surface $\Phi$, through which the change of phase occurs, that constitutes a free boundary and is one of the unknowns of the problem.

With the aid of the usual transformation of Kirchoff we work with the renormalized temperature $\theta$ and make the assumption that the phase change occurs at the fixed temperature $\theta = 0$. Then we can write

$$S = \{\theta < 0\} \quad \text{and} \quad L = \{\theta > 0\} \quad \Phi = \{\theta = 0\}.$$

The strong formulation of the Stefan problem with convection reads (see [R3], [M])

$$\begin{align*}
(\partial_t + \mathbf{v} \cdot \nabla) \theta - \Delta \theta &= 0 \quad \text{in} \quad \{\theta < 0\} \cup \{\theta > 0\}, \\
[\nabla \theta]^+ \cdot \mathbf{n}_x &= -\lambda \mathbf{w} \cdot \mathbf{n}_x = \lambda n_t \quad \text{in} \quad \{\theta = 0\},
\end{align*}$$

where $(\mathbf{n}_x, n_t)$ is the normal vector to $\Phi$ in the space-time cylinder $Q = \Omega \times (0, T)$, $\mathbf{w}$ the velocity of $\Phi$ and $\lambda > 0$ the latent heat. The Stefan condition (2) measures, roughly speaking, the amount of heat used in the phase transition. To (1) and (2) we must add boundary conditions