1. INTRODUCTION

This chapter examines the teaching and learning of linear algebra from the perspective of the different modes of description of vectors and operators. We identify the different modes and their associated language, as used in classroom practice, as well as discuss the mechanisms of translating from one mode to the another. We then focus more specifically on the representation of a vector and an operator relative to a basis in order to illustrate some of the difficulties faced by students of linear algebra.

The teaching of linear algebra at a university level is almost universally regarded as a frustrating experience for instructors and students alike. Many among those who teach such a course have resigned themselves to the fact that this is simply ‘the nature of the beast’ and that not much can be done to change things. This attitude might explain the reason why, until recently, there was a paucity of research work on the learning of linear algebra. Unlike the notions of calculus, which have been researched extensively, most of the research on learning and teaching linear algebra is relatively recent. Starting with the work of Harel (1985), other contributions have been made by, e.g., Robert and Robinet (1989), Rogalski (1990), Dorier (1990), Pavlopoulou (1993), Sierpinska (1995), Dreyfus and Hillel (1998), Sierpinska, Dreyfus and Hillel (1999).

In Canada and the US, linear algebra has traditionally been the first mathematics course that students encounter which is a full-fledged mathematical theory. It is proof-laden, and built systematically from the ground up, with all the fuss about making assumptions explicit, justifying statements by reference to definitions and already proven facts. Therefore, on one level, students’ difficulties with linear algebra stem simply from their inexperience with proofs and proof-based theories. Indeed, students’ proof-related difficulties include: not understanding the need for proofs nor the various proof techniques; not being able to deal with the often implicit quantifiers; confusing necessary and sufficient conditions; making hasty generalizations based on very shaky and sparse evidence. What is perhaps surprising is that this phenomenon is not as local as one might have expected. So, for

---

example, while the teaching of mathematics in France has been a lot more formal than its North American counterpart (or, at least, it was perceived to be), French students of linear algebra do not seem to have any easier time with proofs. Robert and Robinet (1989) asked nearly 380 students in France what they found difficult with the subject, and dealing with proofs, as well as finding that the subject contained too many definitions and new results, were two of the main responses.

The other important aspect of being a mathematical theory is its generality. Knowing linear algebra at this level demands that students start thinking about the objects and operations of algebra, not just in terms of relations between particular matrices, vectors and operators, but in terms of whole structures such as, vector spaces over fields, algebras, and classes of linear operators. Furthermore, students need to be able to appreciate that these structures can be transformed, represented in different ways, and considered as being, or not being, isomorphic. Referring to Piaget and Garcia's (1989) notion of intra-, inter- and trans-level of knowing something, we see that the level in which students need to operate is the trans-level. However, looking at students of linear algebra at our own university, the majority of whom have successfully completed an elementary linear algebra course in their pre-university studies, we can say that their understanding of mathematical objects such as matrices and linear operators can best be characterized as inter-level of thinking (Sierpinska, 1995). Similarly, as in the French study cited above, statements by students that the subject is too abstract and that the general notions such as vector spaces, endomorphisms, bases, dimension and kernel are difficult were quite prominent.

Students' proof-related difficulties are certainly not generic to linear algebra but surface in most undergraduate courses - see, for example, Selden and Selden (1987) regarding Group Theory or Leron (1993) regarding Abstract Algebra. Therefore, in this chapter, we are particularly interested in teasing out those sources of conceptual difficulties that are specific to linear algebra, which include:
- The existence of several languages or modes of description
- The problem of representations
- The applicability of the general theory

2. MODES OF DESCRIPTION IN LINEAR ALGEBRA

A typical course will generally include several modes of description of the basic objects and operations of linear algebra. These modes of description co-exist, are sometimes interchangeable, but are certainly not equivalent. They include:
1. The abstract mode - using the language and concepts of the general formalized theory, including: vector spaces, subspaces, linear span, dimension, operators, kernels
2. The algebraic mode - using the language and concepts of the more specific theory of \( \mathbb{R}^n \), including: \( n \)-tuples, matrices, rank, solutions of systems of equations, row space
3. The geometric mode - using the language and concept of 2- and 3-space, including: directed line segments, points, lines, planes, geometric transformations