In Chapter 5 we have provided a preview of the issues involved in defining homology of maps. In this chapter we revisit this material, but in a rigorous and dimension-independent manner. We begin with the introduction of representable sets. These are sets that can be constructed using elementary cells and represent a larger class than that of cubical sets. This extra flexibility is used in Section 6.2 to construct cubical multivalued maps. As described in Section 5.2, these multivalued maps provide representations of the continuous function for which we wish to compute the homology map. Section 6.3 describes the process by which one passes from the cubical map to a chain map from which one can define a map on homology. Section 6.4 shows that applying the above-mentioned steps (plus perhaps rescaling) to a continuous function leads to a well-defined map on homology. Finally, in the last section, we address the question of when do two different continuous maps give rise to the same map on homology.

6.1 Representable Sets

The goal of this chapter is to define homology maps induced by continuous functions on cubical sets. As mentioned above, an intermediate step is the introduction of cubical multivalued maps. These maps are required to have a weak form of continuity that is most easily expressed using sets defined in terms of elementary cells. For this reason we introduce the notion of a representable set, which is more general than that of a cubical set.

Definition 6.1 A set $Y \subset \mathbb{R}^d$ is representable if it is a finite union of elementary cells. The family of representable sets in $\mathbb{R}^d$ is denoted by $\mathcal{R}^d$.

Example 6.2 The set

$$X = ((0, 1) \times (0, 1)) \cup ([0] \times (0, 1)) \cup ([0, 2] \times [1]) \cup ([3] \times [0])$$
Fig. 6.1. The images generated by the CubTop program applied to the representable set $X$ discussed in Example 6.2. Top left: The image of $X$. The elementary cells in $X$ are displayed either in black (vertices and edges) or in gray (the square). Those in the closure of $X$ but not in $X$ are marked white. Top right: The image of $\text{ch}(X) = \text{cl} X$. Bottom: The image of $\text{oh}(X)$ discussed in Example 6.9.

is representable. It is the union of three elementary vertices,

\[ [0] \times [1], \quad [1] \times [1], \quad [3] \times [0], \]

three elementary edges,

\[ [0] \times (0,1), \quad (0,1) \times [1], \quad (1,2) \times [1], \]

and one elementary square, $(0,1) \times (0,1)$. Figure 6.1, top left, shows the image of $X$ generated by the CubTop program.

We begin with a series of results that characterize representable sets.

**Proposition 6.3** Representable sets have the following properties:

(i) Every elementary cube is representable.
(ii) If $A, B \subseteq \mathbb{R}^d$ are representable, then $A \cup B$, $A \cap B$, and $A \setminus B$ are representable.