Chapter 5

$p$-Groups and Nilpotent Groups

As was mentioned in the introduction of Section 3.2, Sylow's Theorem directs attention to the $p$-subgroups of a finite group. In this chapter we will present some basic facts about $p$-groups (and more generally nilpotent groups), which will be used in later chapters.

In the second section we investigate $p$-groups that contain a cyclic maximal subgroup.

5.1 Nilpotent Groups

A group $G$ is nilpotent, if every subgroup of $G$ is subnormal in $G$. It is evident that this property is equivalent to

$$U < N_G(U) \text{ for every subgroup } U < G.$$ 

As a direct consequence of 1.2.8 on page 14 and Cauchy's Theorem one obtains:

5.1.1 Subgroups and homomorphic images of nilpotent groups are nilpotent. Maximal subgroups of nilpotent groups are normal and of prime index. 

\[\square\]

\[\text{This definition applies only to finite groups. See the footnote on p. 102.}\]
5.1.2 Let $G$ be a group and $Z$ a subgroup of $Z(G)$. Then $G$ is nilpotent if and only if $G/Z$ is nilpotent.

Proof. One direction follows from 5.1.1. For the other direction let $G/Z$ be nilpotent and $U \leq G$. Then $UZ/Z \leq G/Z$ and thus $UZ \leq G$ (1.2.8 on page 14). Since $Z \leq Z(G)$ also $U \leq UZ$. Hence $U \leq G$. 

Result 3.1.10 on page 61 gives the most important class of nilpotent groups:

5.1.3 $p$-Groups are nilpotent.

Recall that $O_p(G)$ denotes the largest normal $p$-subgroup of a group $G$ (3.2.2 on page 63), and $G$ is $p$-closed if $O_p(G)$ is a Sylow $p$-subgroup of $G$.

5.1.4 Theorem. The following statements are equivalent:

(i) $G$ is nilpotent.

(ii) For every $p \in \pi(G)$ $G$ is $p$-closed.

(iii) $G = \prod_{p \in \pi(G)} O_p(G)$.

Proof. (i) $\Rightarrow$ (ii): Let $U := N_G(G_p)$, where $p \in \pi(G)$ and $G_p \in Syl_p G$. Then the Frattini argument yields

$$N_G(U) = U N_{N_G(U)}(G_p) = U$$

since $G_p$ is a Sylow $p$-subgroup of $U$. The definition of nilpotency gives $U = G$ and thus $G_p \leq G$.

(ii) $\Rightarrow$ (iii): This follows from 1.6.5 on page 31.

(iii) $\Rightarrow$ (i): $Z(O_p(G)) \neq 1$ for $p \in \pi(G)$ by 3.1.11 on page 61. Hence also

$$Z(G) = \prod_{p \in \pi(G)} Z(O_p(G)) \neq 1$$

(1.6.2 (a) on page 29). Let $\overline{G} := G/Z(G)$. Then 1.6.2 (c) implies

$$\overline{G} = \prod_{p \in \pi(\overline{G})} \overline{O_p(G)} = \prod_{p \in \pi(\overline{G})} O_p(\overline{G})$$