Chapter 7

Transfer and $p$-Factor Groups

7.1 The Transfer Homomorphism

To search for nontrivial proper normal subgroups is often the first step in the investigation of a finite group. For example, if the group $G$ has such a normal subgroup $N$, then in proofs by induction one frequently gets information about $N$ and $G/N$, allowing one to derive the desired result for $G$ (e.g., 6.1.2 on page 122).

Since normal subgroups are kernels of homomorphisms it is suggestive to construct homomorphisms of $G$ in order to find normal subgroups. The difficulty then is to decide whether the kernel of such a homomorphism is a nontrivial and proper subgroup of $G$.

In the following let $P$ be a subgroup of $G$. In this chapter we define a homomorphism $\tau$ from $G$ into the Abelian group $P/P'$, whose kernel and image can be described by means of $p$-elements if $P$ is a Sylow $p$-subgroup of $G$. This is in the spirit of the philosophy mentioned earlier, that the structure of a group be deduced from its $p$-structure.

If $G$ is non-Abelian, then clearly $\ker \tau$ is nontrivial since $G/\ker \tau$ is Abelian. Hence, either $G$ contains a proper nontrivial normal subgroup or $G = \ker \tau$. In the second case the description of $\ker \tau$ in terms of the conjugacy of $p$-elements in $G$ will yield information concerning the structure of $G$. 

H. Kurzweil et al., The Theory of Finite Groups
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Let
\[ \overline{P} := P/P' \]
be the commutator factor group of \( P \) and

\[ P \to \overline{P} \quad \text{with} \quad x \mapsto \overline{x} \]

the natural epimorphism to the *Abelian* group \( \overline{P} \).

Let \( S \) be the set of transversals of \( P \) in \( G \). For \( R, S \in S \) let

\[ R|S := \prod_{(r,s) \in R \times S, \overline{r}_r = \overline{P}_s} r_s^{-1} \quad (\in \overline{P}). \]

(Compare with the definition on page 71.) Since the factors are elements of the Abelian group \( \overline{P} \) this product does not depend on their ordering. As in Section 3.3 for \( R, S, T \in S \) the following properties hold:

1. \((R|S)^{-1} = S|R\)
2. \((R|S) (S|T) = R|T.\)

We investigate the action of \( G \) on \( S \) by right multiplication:

\[ S \xrightarrow{g \in G} Sg. \]

Then

3. \( Rg \mid Sg = R|S \)

and

4. \( Rg \mid R = Sg \mid S. \)

For the proof of (4) note that

\[ (Rg|R) (Sg|S)^{-1} = (Rg|R) (R|Sg)(R|Sg)^{-1} (Sg|S)^{-1} \]

\[ = (Rg|R) (R|Sg) ((R|Sg) (Sg|S))^{-1} \]

\[ \stackrel{(2)}{=} (Rg|Sg) (R|S)^{-1} \stackrel{(3)}{=} 1. \]

### 7.1.1 Transfer Homomorphism.

*Let \( S \in S. \) The mapping*

\[ \tau_{G \to \overline{P}} : G \to \overline{P} \quad \text{with} \quad g \mapsto Sg|S \]

*is a homomorphism that is independent of the choice of \( S \in S. \)*