According to Galois theory\(^1\) a polynomial equation is solvable by radicals iff the associated Galois group is solvable. For example, since the alternating group \(A_5\) is simple (cf. Problem 6 of Section 17), there is no root formula for a quintic with Galois group \(A_5\).

Since the symmetry group of the icosahedron is (isomorphic to) \(A_5\) (Section 17), the question arises naturally whether there is any connection between the icosahedron and the solutions of quintic equations. This is the subject of Klein's famous *Icosahedron Book*.\(^2\)

We devote this (admittedly long) section to sketch Klein's main result. We will treat the material here somewhat differently than Klein, and rely more on geometry. Unlike the *Icosahedron Book*,

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\(^1\) From now on, we will use some basic facts from Galois theory. For a quick summary, see Appendix F.

we will take as direct a path to the core results as possible. Due to the complexity of the exposition, we divide our treatment into subsections.

\section{Polyhedral Equations}

In Section 24, we defined, for each finite Möbius group $G$, a $G$-invariant rational function $q : \mathbb{C} \to \mathbb{C}$. Geometrically, the extension $q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the analytic projection of a branched covering between Riemann spheres, and the branch values are $w = 0$, $w = 1$, and $w = \infty$, with branch numbers $\nu_2$, $\nu_1$, and $\nu_0$ minus one. (In the case of the dihedron $\nu_0$ and $\nu_2$ are switched.) As in the proof of Theorem 18, for a given $w \in \mathbb{C}$, the equation $q(z) = w$ for $z$ can be written as a degree-$|G|$ polynomial equation

$$P(z) - wQ(z) = 0,$$

where $q = P/Q$ with $P$ and $Q$ the polynomial numerator and denominator of $q$ (with no common factors). We call this the \textit{polyhedral equation associated to} $G$. Clearly, this polynomial equation has $|G|$ solutions (counted with multiplicity, and depending on the parameter $w$). We now consider this equation for each $G$. The case of the cyclic group $C_n$ is obvious, since the associated equation is $z^n = w$, and the solutions are simply the $n$th roots of $w$. Using the second table in Section 24, for the equation of the dihedron we have

$$q_{D_n}(z) = -\frac{\alpha(z_1, z_2)^2}{\gamma(z_1, z_2)^n} = -\frac{(z_1^n - z_2^n)^2}{4z_1^n z_2^n} = -\frac{(z^n - 1)^2}{4z^n} = w,$$

where we have indicated the acting group by a subscript. Multiplying out, we obtain the equation of the dihedron, a quadratic equation in $z^n$. This can easily be solved:

$$z = q_{D_n}^{-1}(w) = \sqrt[4]{1 - 2w \pm 2\sqrt{w(w - 1)}}.$$

Inverting $q_{D_n}$ amounts to extracting a square root followed by the extraction of an $n$th root.