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The Square Root of 2

To arouse interest in Pell's equation and introduce some of the ideas that will be important in its study, we will examine the question of the irrationality of the square root of 2. This has its roots in Greek mathematics and can be looked at from the standpoint of arithmetic, geometry, or analysis.

The standard arithmetical argument for the irrationality of 2 goes like this. Suppose that \( \sqrt{2} \) is rational. Then we can write it as a fraction whose numerator and denominator are positive integers. Let this fraction be written in lowest terms: \( \sqrt{2} = \frac{p}{q} \), with the greatest common divisor of \( p \) and \( q \) equal to 1. Then \( p^2 = 2q^2 \), so that \( p \) must be even. But then \( p^2 \) is divisible by 4, which means that \( q \) must be even along with \( p \). Since this contradicts our assumption that the fraction is in lowest terms, we must abandon our supposition that the square root of 2 is rational.

Another way of looking at this argument is to note that if \( \sqrt{2} = \frac{p}{q} \) as above, then, since numerator and denominator are both even, we can write it as a fraction with strictly smaller numerator and denominator, and can continue doing this indefinitely. This is impossible. This “descent” approach has an echo in a geometric argument given below. This will, in turn, bring into play the role of recursions.

1.1 Can the Square Root of 2 Be Rational?

Two alternative arguments, one in Exercise 1.1 and the other in Exercises 1.2–1.4, are presented. The second argument introduces two sequences, defined recursively, that figure in solutions to the equation \( x^2 - 2y^2 = \pm 1 \).

Exercise 1.1. An attractive argument that \( \sqrt{2} \) is not a rational number utilizes the geometry of the square. Let \( ABCD \) be a square with diagonal \( AC \).

(a) Determine a point \( E \) on \( AC \) for which \( AE = BC \). Let \( F \) be a point selected on \( BC \) such that \( EF \perp AC \). Prove that \( BF = FE \).

(b) Complete the square \( CEFG \). Suppose that the lengths of the side and diagonal of square \( ABCD \) are \( a_1 \) and \( b_1 \), respectively. Prove that the lengths of the side and diagonal of the square \( CEFG \) are \( b_1 - a_1 \) and \( 2a_1 - b_1 \), respectively.
(c) We can perform the same construction on square $CEF G$ to produce an even smaller square and so continue indefinitely. Suppose that the sides of the successive squares have lengths $a_1, a_2, a_3, \ldots$, and their corresponding diagonals have lengths $b_1, b_2, b_3, \ldots$. Argue that both sequences are decreasing and that they jointly satisfy the recursion relations

$$a_n = b_{n-1} - a_{n-1},$$
$$b_n = 2a_{n-1} - b_{n-1},$$

for $n \geq 1$.

(d) Suppose that the ratio $b_1/a_1$ of the diagonal and side lengths of the square $ABCD$ is equal to the ratio $p/q$ of positive integers $p$ and $q$. This means that there is a length $\lambda$ for which $a_1 = q\lambda$ and $b_1 = p\lambda$ (or as the Greeks might have put it, the length $\lambda$ “measures” both the side and diagonal of the square). Verify that $a_2 = (p - q)\lambda$ and $b_2 = (2q - p)\lambda$.

(e) Prove that for each positive integer $n$, both $a_n$ and $b_n$ are positive integer multiples of $\lambda$.

(f) Arithmetically, observe that there are only finitely many pairs smaller than $(p, q)$ that can serve as multipliers of $\lambda$ when we construct smaller squares as described. Deduce that only finitely many squares are thus constructible.

Geometrically, note that we can repeat the process as often as desired. Now complete the contradiction argument and deduce that the ratio of the lengths of the diagonal and side of a square is not rational. ♦

We can pick up the recursion theme in another way, in this case getting a sequence of pairs of integers that increase rather than decrease in size and whose ratio will