The orthogonality of a pair of subspaces was defined and discussed in Chapter 12. There are pairs of subspaces that are not orthogonal but that have a weaker property called essential disjointness—orthogonal subspaces are essentially disjoint, but essentially disjoint subspaces are not necessarily orthogonal. Unlike orthogonality, essential disjointness does not depend on the choice of inner product. The essential disjointness of a pair of subspaces is defined and discussed in the present chapter. The concept of essential disjointness arises in a very natural and fundamental way in results (like those of Sections 17.2, 17.3, and 17.5) on the ranks, row and column spaces, and generalized inverses of partitioned matrices and of sums and products of matrices.

It was found in Chapter 12 that a subspace $U$ of a linear space $V$ and its orthogonal complement $U^\perp$ (which is a subspace of $V$ that is by definition orthogonal to $U$ and for which $\dim U + \dim U^\perp = \dim V$) have the property that every matrix $Y$ in $V$ can be uniquely expressed as the sum of a matrix in $U$ (which is the projection of $Y$ on $U$) and a matrix in $U^\perp$ (which is the projection of $Y$ on $U^\perp$)—refer to Theorem 12.5.11. It is shown in Section 17.6 that this property extends to any subspaces $U$ and $W$ of $V$ that are essentially disjoint and that have $\dim U + \dim W = \dim V$. That is, for any such subspaces $U$ and $W$, every matrix in $V$ can be uniquely expressed as the sum of a matrix in $U$ and a matrix in $W$.

### 17.1 Definitions and Some Basic Properties

Let $U$ and $V$ represent subsets of the linear space $\mathbb{R}^{m\times n}$ of all $m \times n$ matrices. The intersection of $U$ and $V$ is the subset (of $\mathbb{R}^{m\times n}$) comprising all $m \times n$ matrices that are common to $U$ and $V$ (i.e., that belong to both $U$ and $V$) and is denoted by the symbol $U \cap V$.

If $U$ and $V$ are subspaces (of $\mathbb{R}^{m\times n}$), then their intersection $U \cap V$ is also a subspace. To see this, suppose that $U$ and $V$ are subspaces. Then, the $m \times n$ null matrix $0$ belongs to both $U$ and $V$ and hence to $U \cap V$, so that $U \cap V$ is nonempty. Further, for any matrix $A$ in $U \cap V$ and any scalar $k$, $A$ is in $U$ and
also in \( \mathcal{U} \), implying that \( kA \) is in both \( \mathcal{U} \) and \( \mathcal{V} \) and consequently in \( \mathcal{U} \cap \mathcal{V} \). And, for any matrices \( A \) and \( B \) in \( \mathcal{U} \cap \mathcal{V} \), both \( A \) and \( B \) are in \( \mathcal{U} \) and both are also in \( \mathcal{V} \), implying that \( A + B \) is in \( \mathcal{U} \) and also in \( \mathcal{V} \) and hence that \( A + B \) is in \( \mathcal{U} \cap \mathcal{V} \).

More generally, the intersection of \( k \) subsets \( \mathcal{U}_1, \ldots, \mathcal{U}_k \) of \( \mathbb{R}^{m \times n} \) (where \( k \geq 2 \)) is the subset (of \( \mathbb{R}^{m \times n} \)) comprising all \( m \times n \) matrices that are common to \( \mathcal{U}_1, \ldots, \mathcal{U}_k \) and is denoted by the symbol \( \mathcal{U}_1 \cap \cdots \cap \mathcal{U}_k \) (or alternatively by \( \cap_{i=1}^k \mathcal{U}_i \) or simply \( \cap \mathcal{U}_j \)). And, if \( \mathcal{U}_1, \ldots, \mathcal{U}_k \) are subspaces (of \( \mathbb{R}^{m \times n} \)), then \( \mathcal{U}_1 \cap \cdots \cap \mathcal{U}_k \) is also a subspace.

Note that if (for subsets \( \mathcal{U}, \mathcal{V} \), and \( \mathcal{W} \) of \( \mathbb{R}^{m \times n} \)) \( \mathcal{W} \subseteq \mathcal{U} \) and \( \mathcal{W} \subseteq \mathcal{V} \), then \( \mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V} \). Note also that if \( \mathcal{U} \subseteq \mathcal{V} \) and \( \mathcal{V} \subseteq \mathcal{W} \), then \( \mathcal{U} \subseteq \mathcal{W} \cap \mathcal{V} \).

The sum of two (nonempty) subsets \( \mathcal{U} \) and \( \mathcal{V} \) of \( \mathbb{R}^{m \times n} \) is defined to be the subset \( \{ U + V : U \in \mathcal{U}, V \in \mathcal{V} \} \) comprising every \( m \times n \) matrix that is expressible as the sum of a matrix in \( \mathcal{U} \) and a matrix in \( \mathcal{V} \). It is denoted by the symbol \( \mathcal{U} + \mathcal{V} \).

Note that, except for special cases, \( \mathcal{U} + \mathcal{V} \) differs from the union \( \mathcal{U} \cup \mathcal{V} \) of \( \mathcal{U} \) and \( \mathcal{V} \), which by definition is the subset (of \( \mathbb{R}^{m \times n} \)) comprising all \( m \times n \) matrices that belong to \( \mathcal{U} \) or \( \mathcal{V} \).

The following lemma provides a useful characterization of the sum of two subspaces of \( \mathbb{R}^{m \times n} \).

**Lemma 17.1.1.** Let \( \mathcal{U} \) and \( \mathcal{V} \) represent subspaces of \( \mathbb{R}^{m \times n} \). If \( \mathcal{U} = \{ 0 \} \) (i.e., if there are no nonnull matrices in \( \mathcal{U} \)), then \( \mathcal{U} + \mathcal{V} = \mathcal{U} \), and similarly if \( \mathcal{V} = \{ 0 \} \), then \( \mathcal{U} + \mathcal{V} = \mathcal{V} \). Further, if \( \mathcal{U} \) is spanned by a (finite nonempty) set of \( m \times n \) matrices \( \mathcal{U}_1, \ldots, \mathcal{U}_s \) and \( \mathcal{V} \) is spanned by a set of matrices \( \mathcal{V}_1, \ldots, \mathcal{V}_t \), then

\[
\mathcal{U} + \mathcal{V} = \text{sp}(\mathcal{U}_1, \ldots, \mathcal{U}_s, \mathcal{V}_1, \ldots, \mathcal{V}_t).
\]

**Proof.** That \( \mathcal{U} + \mathcal{V} = \mathcal{U} \) if \( \mathcal{V} = \{ 0 \} \) and that \( \mathcal{U} + \mathcal{V} = \mathcal{V} \) if \( \mathcal{U} = \{ 0 \} \) is obvious. Suppose now that \( \mathcal{U} \) is spanned by the set \( \{ \mathcal{U}_1, \ldots, \mathcal{U}_s \} \) and \( \mathcal{V} \) by the set \( \{ \mathcal{V}_1, \ldots, \mathcal{V}_t \} \). Then, for any matrix \( U \) in \( \mathcal{U} \) and any matrix \( V \) in \( \mathcal{V} \), there exist scalars \( c_1, \ldots, c_s \) and \( k_1, \ldots, k_t \) such that \( U = \sum_{i=1}^s c_i \mathcal{U}_i \) and \( V = \sum_{j=1}^t k_j \mathcal{V}_j \), and hence such that

\[
U + V = \sum_{i=1}^s c_i \mathcal{U}_i + \sum_{j=1}^t k_j \mathcal{V}_j,
\]

implying that \( U + V \in \text{sp}(\mathcal{U}_1, \ldots, \mathcal{U}_s, \mathcal{V}_1, \ldots, \mathcal{V}_t) \). Thus,

\[
\mathcal{U} + \mathcal{V} \subset \text{sp}(\mathcal{U}_1, \ldots, \mathcal{U}_s, \mathcal{V}_1, \ldots, \mathcal{V}_t). \tag{1.1}
\]

Further, for any matrix \( A \) in \( \text{sp}(\mathcal{U}_1, \ldots, \mathcal{U}_s, \mathcal{V}_1, \ldots, \mathcal{V}_t) \), there exist scalars \( c_1, \ldots, c_s, k_1, \ldots, k_t \) such that \( A = \sum_{i=1}^s c_i \mathcal{U}_i + \sum_{j=1}^t k_j \mathcal{V}_j \), implying that \( A = U + V \) for some matrix \( U \) in \( \mathcal{U} \) and some matrix \( V \) in \( \mathcal{V} \) (namely, \( U = \sum_{i=1}^s c_i \mathcal{U}_i \) and \( V = \sum_{j=1}^t k_j \mathcal{V}_j \)). It follows that \( \text{sp}(\mathcal{U}_1, \ldots, \mathcal{U}_s, \mathcal{V}_1, \ldots, \mathcal{V}_t) \subset \mathcal{U} + \mathcal{V} \), leading (in light of result (1.1)) to the conclusion that \( \mathcal{U} + \mathcal{V} = \text{sp}(\mathcal{U}_1, \ldots, \mathcal{U}_s, \mathcal{V}_1, \ldots, \mathcal{V}_t) \). Q.E.D.

As an immediate consequence of Lemma 17.1.1, we have the following corollary.