In this chapter, we study in detail the relationships among geodesics, lengths, and distances on a Riemannian manifold. A primary goal is to show that all length-minimizing curves are geodesics, and that all geodesics are length minimizing, at least locally. A key ingredient in the proofs is the symmetry of the Riemannian connection. Later in the chapter, we study the property of geodesic completeness, which means that all maximal geodesics are defined for all time, and prove the Hopf–Rinow theorem, which states that a Riemannian manifold is geodesically complete if and only if it is complete as a metric space.

Throughout this chapter, \( M \) is a smooth \( n \)-manifold endowed with a fixed Riemannian metric \( g \). All covariant derivatives and geodesics are understood to be with respect to the Riemannian connection of \( g \).

Most of the results of this chapter do not apply to pseudo-Riemannian metrics, at least not without substantial modification. For a treatment of lengths of curves in the pseudo-Riemannian setting, see [O’N83].

Lengths and Distances on Riemannian Manifolds

We are now in a position to introduce two of the most fundamental concepts from classical geometry into the Riemannian setting: lengths of curves and distances between points. We begin with lengths.
Lengths of Curves

If $\gamma: [a, b] \to M$ is a curve segment, we define the length of $\gamma$ to be

$$L(\gamma) := \int_a^b |\dot{\gamma}(t)| \, dt.$$ 

Sometimes, for the sake of clarity, we emphasize the dependence on the metric by using the notation $L_g$ instead of $L$.

The key feature of the length of a curve is that it is independent of parametrization. To make this notion precise, we define a reparametrization of $\gamma$ to be a curve segment of the form $\tilde{\gamma} = \gamma \circ \varphi$, where $\varphi: [c, d] \to [a, b]$ is a smooth map with smooth inverse. We say it is a forward reparametrization if $\varphi$ is orientation preserving, and a backward reparametrization if not.

**Lemma 6.1.** For any curve segment $\gamma: [a, b] \to M$, and any reparametrization $\tilde{\gamma}$ of $\gamma$, $L(\gamma) = L(\tilde{\gamma})$.

**Exercise 6.1.** Prove Lemma 6.1.

For measuring distances between points, it is useful to modify slightly the class of curves we consider. A regular curve is a smooth curve $\gamma: I \to M$ such that $\dot{\gamma}(t) \neq 0$ for $t \in I$. Intuitively, this prevents the curve from having "cusps" or "kinks." More formally, because the tangent vector $\dot{\gamma}(t)$ is the push-forward $\gamma_* (d/dt)$, a regular curve is an immersion of the interval $I$ into $M$. (If $I$ has one or two endpoints, it has to be considered as a manifold with boundary.) Note that geodesics are automatically regular, since they have constant speed.

A continuous map $\gamma: [a, b] \to M$ is called a piecewise regular curve segment if there exists a finite subdivision $a = a_0 < a_1 < \cdots < a_k = b$ such that $\gamma|_{[a_{i-1}, a_i]}$ is a regular curve for $i = 1, \ldots, k$. All distances on a Riemannian manifold will be measured along such curve segments. For brevity, we refer to a piecewise regular curve segment as an admissible curve. It's also convenient to allow a trivial constant curve $\gamma: \{a\} \to M, \gamma(a) = p$, to be considered an admissible curve.

The definition implies that an admissible curve must have well-defined, nonzero, one-sided velocity vectors when approaching $a_i$ from either side, but the two limiting velocity vectors need not be equal. We denote these one-sided velocities by

$$\dot{\gamma}(a^-_i) := \lim_{t \to a^-_i} \dot{\gamma}(t);$$
$$\dot{\gamma}(a^+_i) := \lim_{t \to a^+_i} \dot{\gamma}(t).$$

Let $\gamma: [a, b] \to M$ be an admissible curve, and $a = a_0 < a_1 < \cdots < a_k = b$ a subdivision as above. The length of $\gamma$ is defined simply as the sum of the lengths of the smooth subsegments $\gamma|_{[a_{i-1}, a_i]}$. We can broaden