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Two-Phase Incompressible Flow

21.1 Introduction

The earliest real success in the coupling of the level set method to problems involving external physics came in computing two-phase incompressible flow, in particular see the work of Sussman et al. [160] and Chang et al. [38]. In two-phase incompressible flow, the Navier-Stokes equations are used to model the fluids on both sides of the interface. Generally, the fluids have different densities and viscosities, so these quantities are discontinuous across the interface. Both the discontinuous viscosity and surface tension forces cause the pressure to be discontinuous across the interface as well. In addition, a discontinuous viscosity leads to kinks in the velocity field across the interface.

In [132], Peskin introduced the “immersed boundary” method for simulating an elastic membrane immersed in an incompressible fluid flow. This method uses a δ-function formulation to smear out the numerical solution in a thin region about the immersed interface. This concept has been used by a variety of authors to solve a number of interface-related problems. For example, [160] defined a numerically smeared-out density and viscosity as functions of the level set function,

\begin{align}
\rho(\phi) &= \rho^- + (\rho^+ - \rho^-) H(\phi), \\
\mu(\phi) &= \mu^- + (\mu^+ - \mu^-) H(\phi),
\end{align}

(21.1) (21.2)

where \( H(\phi) \) is the numerically smeared-out Heaviside function defined by equation (1.22). This removes all discontinuities across the interface, except
the jump in pressure due to surface tension, \( [p] = \sigma \kappa \), where \( \sigma \) is a constant coefficient and \( \kappa \) is the local curvature of the interface. Using the immersed boundary method to smear out the pressure across the interface leads to continuity of the pressure, \( [p] = 0 \), and loss of all surface-tension effects. This was remedied by Brackbill et al. [18] in the context of volume of fluid (VOF) methods and by Unverdi and Tryggvason [168] in the context of front-tracking methods by adding a new forcing term to the right-hand side of the momentum equations. In the context of level set methods (see [160]) this new forcing term takes the form

\[
\frac{\delta(\phi)\sigma \kappa \vec{N}}{\rho},
\]

(21.3)

where \( \delta(\phi) \) is the smeared-out delta function given by equation (1.23). In the spirit of the immersed boundary method, [18] referred to this as the continuum surface force (CSF) method.

In the interest of solving for the pressure jump directly, Liu et al. [106] devised a new boundary-condition-capturing approach for the variable-coefficient Poisson equation to solve problems of the form

\[
\nabla \left( \frac{1}{\rho} \nabla p \right) = f,
\]

(21.4)

where the jump conditions \( [p] = g \) and \([ (1/\rho) \nabla p \cdot \vec{N} ] = h \) are given. Here, \( \rho \) can be discontinuous across the interface as well. Figure 21.1 shows a typical example of the discontinuous solutions obtained using this method. Note that both the pressure and its derivatives are clearly discontinuous across the interface. Kang et al. [91] applied this technique to multiphase incompressible flow, illustrating the ability to solve these equations without smearing out the density, the viscosity, or the pressure across the interface. Moreover, the \( \delta(\phi)\sigma \kappa \vec{N}/\rho \) forcing term was not needed, since the pressure jump was modeled directly. Figure 21.2 shows a water drop falling through the air into the water. Here, surface tension forces cause the spherically shaped region at the top of the resulting water jet in the last frame of the figure.

LeVeque and Li [102] proposed a second-order accurate sharp interface method to solve equation (21.4). In general, one needs to solve a linear system of equations of the form \( A \vec{p} = \vec{b} \), where \( \vec{p} \) are the unknown pressures, \( A \) is the coefficient matrix, and \( \vec{b} \) is the right-hand side. Unfortunately, the discretization in [102] leads to a complicated asymmetric coefficient matrix, making this linear system difficult to solve. So far, this method has not been applied to two-phase incompressible flow. In contrast, the discretization proposed in [106] leads to a symmetric coefficient matrix identical to the standard one obtained when both the pressure and its derivatives are continuous across the interface. Adding the jump conditions only requires modification of the right-hand side, \( \vec{b} \). This allows the use of standard