7

Exponential of a Matrix, Polar Decomposition, and Classical Groups

7.1 The Polar Decomposition

The polar decomposition of matrices is defined by analogy with that in the complex plane: If \( z \in \mathbb{C}^* \), there exists a unique pair \((r, q) \in (0, +\infty) \times S^1\) (\(S^1\) denotes the unit circle, the set of complex numbers of modulus 1) such that \( z = rq \). If \( z \) acts on \( \mathbb{C} \) (or on \( \mathbb{C}^* \)) by multiplication, this action can be decomposed as the product of a rotation of angle \( \theta \) (where \( q = \exp(i\theta) \)) with a homothety of ratio \( r > 0 \). The fact that these two actions commute is a consequence of the commutativity of the multiplicative group \( \mathbb{C}^* \); this property does not hold for the polar decomposition in \( \text{GL}_n(k) \), \( k = \mathbb{R} \) or \( \mathbb{C} \), because the general linear group is not commutative.

Let us recall that \( \text{HPD}_n \) denotes the (open) cone of matrices of \( \text{M}_n(\mathbb{C}) \) that are Hermitian positive definite, while \( \text{U}_n \) denotes the group of unitary matrices. In \( \text{M}_n(\mathbb{R}) \), \( \text{SPD}_n \) is the set of symmetric positive definite matrices, and \( \text{O}_n \) is the orthogonal group. The group \( \text{U}_n \) is compact, since it is closed and bounded in \( \text{M}_n(\mathbb{C}) \). Indeed, the columns of unitary matrices are unit vectors, so that \( \text{U}_n \) is bounded. On the other hand, \( \text{U}_n \) is defined by an equation \( U^*U = I_n \), where the map \( U \mapsto U^*U \) is continuous; hence \( \text{U}_n \) is closed. By the same arguments, \( \text{O}_n \) is compact.

Polar decomposition is a fundamental tool in the theory of finite-dimensional Lie groups and Lie algebras. For this reason, it is intimately related to the exponential map. We shall not consider these two notions here in their full generality, but we shall restrict attention to their matricial aspects.
Theorem 7.1.1 For every $M \in \text{GL}_n(C)$, there exists a unique pair $(H, Q) \in \text{HPD}_n \times U_n$

such that $M = HQ$. If $M \in \text{GL}_n(R)$, then $(H, Q) \in \text{SPD}_n \times O_n$.

The map $M \mapsto (H, Q)$, called the polar decomposition of $M$, is a homeomorphism between $\text{GL}_n(C)$ and $\text{HPD}_n \times U_n$ (respectively between $\text{GL}_n(R)$ and $\text{SPD}_n \times O_n$).

Theorem 7.1.2 Let $H$ be a positive definite Hermitian matrix. There exists a unique positive definite Hermitian matrix $h$ such that $h^2 = H$. If $H$ is real-valued, then so is $h$. The matrix $h$ is called the square root of $H$, and is denoted by $h = \sqrt{H}$.

Proof

We prove Theorem 7.1.1 and obtain Theorem 7.1.2 as a by-product.

Existence. Since $MM^* \in \text{HPD}_n$, we can diagonalize $MM^*$ by a unitary matrix

$$MM^* = U^*DU, \quad D = \text{diag}(d_1, \ldots, d_n),$$

where $d_j \in (0, +\infty)$. The matrix $H := U^* \text{diag}(\sqrt{d_1}, \ldots, \sqrt{d_n})U$ is Hermitian positive definite and satisfies $H^2 = HH^* = MM^*$. Then $Q := H^{-1}M$ satisfies $Q^*Q = M^*H^{-2}M = M^*(MM^*)^{-1}M = I_n$, hence $Q \in U_n$. If $M \in \text{M}_n(R)$, then clearly $MM^*$ is real symmetric. In fact, $U$ is orthogonal and $H$ is real symmetric. Hence $Q$ is real orthogonal. Note: $H$ is called the square root of $MM^*$.

Uniqueness. Let $M = H'Q'$ be another suitable decomposition. Then $N := H^{-1}H' = Q(Q')^{-1}$ is unitary, so that $\text{Sp}(N) \subset S^1$. Let $S \in \text{HPD}_n$ be a positive definite Hermitian square root of $H'$ (we shall prove below that it is unique). Then $N$ is similar to $N' := SH^{-1}S$. However, $N' \in \text{HPD}_n$. Hence $N$ is diagonalizable, with real positive eigenvalues. Hence $\text{Sp}(N) = \{1\}$, and $N$ is therefore similar, and thus equal, to $I_n$.

This proves that the positive definite Hermitian square root of a matrix of $\text{HPD}_n$ is unique in $\text{HPD}_n$, since otherwise, our construction would provide several polar decompositions. We have thus proved Theorem 7.1.2 in passing.

Smoothness. The map $(H, Q) \mapsto HQ$ is polynomial, hence continuous. Conversely, it is enough to prove that $M \mapsto (H, Q)$ is sequentially continuous, since $\text{GL}_n(C)$ is a metric space. Let $(M_k)_{k \in \mathbb{N}}$ be a convergent sequence in $\text{GL}_n(C)$ and let $M$ be its limit. Let us denote by $M_k = H_kQ_k$ and $M = HQ$ their respective polar decompositions. Let $R$ be a cluster point of the sequence $(Q_k)_{k \in \mathbb{N}}$, that is, the limit of some subsequence $(Q_{k_l})_{l \in \mathbb{N}}$, with $k_l \to +\infty$. Then $H_{k_l} = M_{k_l}Q_{k_l}^*$ converges to $S := MR^*$. The matrix $S$ is Hermitian positive semidefinite (because it is the limit of the $H_{k_l}$’s) and invertible (because it is the product of $M$ and $R^*$). It is thus positive definite. Hence, $SR$ is a polar decomposition of $M$. The uniqueness part ensures that $R = Q$ and $S = H$. The sequence $(Q_k)_{k \in \mathbb{N}}$,