Quality Control for Products with Warranty

The objective of this chapter is to develop an inspection procedure for end products that are supplied to customers with some type of warranty (or service contract), which obliges the manufacturer to provide repair, replacement, or, in some cases, refund to the consumer for a product that has failed within a certain period of time as specified by the contract.

Consider the context of batch manufacturing. Suppose we are dealing with a batch of $N$ units and the quality of the batch is characterized by $\Theta$, the proportion of defective units. Assume $\Theta$ is only known through its distribution. Both defective and nondefective units have random lifetimes with given distributions.

Suppose there is an inspection-repair procedure that can identify and repair all the defective units. Hence, if we follow a 100% inspection, we can guarantee that all $N$ units are nondefective before they are shipped. However, inspection and repair do not come free. At the least, they will consume production capacity. The essence of the problem here is to strike a balance between the warranty cost and the inspection-repair cost.

The policy we shall identify and prove to be optimal has a simple, sequential structure: it is characterized by a sequence of threshold values, $d_{n_0} \leq \ldots \leq d_n \leq \ldots \leq d_{n_1}$, such that if $D_n$ denotes the number of defective units among $n$ inspected units, then the optimal policy is to stop inspection at the first $n$ that satisfies $D_n < d_n$.

The key that underlies the optimality of this policy is a simple and intuitive monotone property: the higher the defective rate—not in terms of $\Theta$, but in terms of its posterior estimate, given the outcome of the inspection—the more inspection an optimal policy will require. It turns out that this
monotone property is a direct consequence of the warranty cost, as a function of the number of inspected units and the conditional defective rate, satisfying a so-called $K$-submodularity property, which is a strengthening of the usual notion of submodularity. Based on this property, we are able to identify several structural results of the optimal policy and eventually characterize the policy itself in terms of certain simple thresholds.

In §3.1 we spell out the precise details of the problem. We then elaborate on the $K$-submodularity property of the expected warranty cost in §3.2 and related properties of the conditional distribution of the defective rate in §3.3. The optimal control problem is formulated in §3.4, where several key structural properties of the optimal policy are established in Theorems 3.11, 3.12, and 3.14, which lead to a statement of the optimal policy in Theorem 3.16. A special case, the individual warranty model, is studied in §3.5. Two numerical examples and possible extensions are presented in §3.6.

### 3.1 Warranty Cost Functions

A batch of $N$ units of a certain product has been completed on the production line. The units will supply customer demand, under some kind of warranty that will be specified later. We want to devise an inspection-repair procedure to ensure quality and to balance inspection-repair cost on the one hand and warranty cost on the other.

Assume each unit in the batch of $N$ is either defective or nondefective. A nondefective unit has a lifetime of $X$, and a defective unit has a lifetime of $Y$. Both $X$ and $Y$ are random variables. Suppose $X$ and $Y$ are ordered under stochastic ordering, $X \geq_{st} Y$, i.e.,

$$P[X \geq a] \geq P[Y \geq a] \text{ for all } a \geq 0.$$

(Refer to Definition 2.1.)

Assume an inspection procedure can identify whether a unit is defective at a cost of $c_i$ per unit. Each defective unit identified by the inspection is repaired, at a cost of $c_r$ per unit, and becomes a nondefective unit.

The quality of the batch, before any inspection and repair, is represented by $\Theta$, the proportion of defective units in the batch. Here $\Theta$ is assumed to be a random variable, with a known distribution function. Without loss of generality, assume $\Theta \in [\theta_0, \theta_1]$, where $\theta_0$ and $\theta_1$ are two given constants, $0 \leq \theta_0 \leq \theta_1 \leq 1$. (This essentially follows the quality model of Marner [58].) Note that letting $\theta_0 = \theta_1 = \theta$ models the special case of a deterministic $\Theta = \theta$. However, this special case restricts the number of defectives in $n$ items to a binomial distribution, with a squared coefficient of variation equal to $(1 - \theta)/(n\theta)$, much too small—when $n$ is large—for many applications.

Because there is no a priori discernible information about the quality of any units in the batch, we assume that each inspection will identify a