Chapter 14

FUZZY MEASURES AND INTEGRALS IN MCDA

Michel Grabisch
Université Paris I – Panthéon-Sorbonne
LIP6, 8 rue du Capitaine Scott
75015 Paris, France
michel.grabisch@lip6.fr

Christophe Labreuche
Thales Research & Technology
Domaine de Corbeville
91404 Orsay Cedex, France
christophe.labreuche@thalesgroup.com

Abstract
This chapter aims at a unified presentation of various methods of MCDA based on fuzzy measures (capacity) and fuzzy integrals, essentially the Choquet and Sugeno integral. A first section sets the position of the problem of multicriteria decision making, and describes the various possible scales of measurement (cardinal unipolar and bipolar, and ordinal). Then a whole section is devoted to each case in detail: after introducing necessary concepts, the methodology is described, and the problem of the practical identification of fuzzy measures is given. The important concept of interaction between criteria, central in this chapter, is explained in detail. It is shown how it leads to k-additive fuzzy measures. The case of bipolar scales leads to the general model based on bi-capacities, encompassing usual models based on capacities. A general definition of interaction for bipolar scales is introduced. The case of ordinal scales leads to the use of Sugeno integral, and its symmetrized version when one considers symmetric ordinal scales. A practical methodology for the identification of fuzzy measures in this context is given.

Keywords: Choquet integral, fuzzy measure, interaction, bi-capacities.
### 1. Introduction

MultiCriteria Decision Aid (MCDA) aims at modeling the preferences of a Decision Maker (DM) over alternatives described by several points of view, which are denoted by \(X_1, \ldots, X_n\). An alternative is characterized by a value w.r.t. each point of view and is thus identified with a point in the Cartesian product \(X\) of the points of view: \(X = X_1 \times \cdots \times X_n\). We denote by \(N := \{1, \ldots, n\}\) the index set of points of view. The preference relation of the DM over alternatives is denoted by \(\succeq\). For \(x, y \in X\), “\(x \succeq y\)” means that the DM prefers alternative \(x\) to \(y\).

The main concern in practice is to come up with the knowledge of \(\succeq\) on \(X \times X\) from a relatively small amount of questions asked to the DM on \(\succeq\). The information provided by the DM can be composed of examples of comparisons between alternatives, which gives \(\succeq\) on a subset of \(X \times X\), as well as more qualitative judgments, whose modelling is more complex, and depends on the kind of representation of \(\succeq\) we choose. In general, we look for a numerical representation [43] \(u : X \rightarrow \mathbb{R}\) such that:

\[
\forall x, y \in X, \quad x \succeq y \iff u(x) \geq u(y).
\]  
(14.1)

It is classical to write \(u\) in the following way [42]:

\[
u(x) = F(u_1(x_1), \ldots, u_n(x_n)) \quad \forall x \in X,
\]  
(14.2)

where the \(u_i\)'s : \(X_i \rightarrow \mathbb{R}\) are called the utility functions and \(F : \mathbb{R}^n \rightarrow \mathbb{R}\) is an aggregation function. A result by Krantz et al. gives the axioms that characterize the representation of \(\succeq\) by (14.2) [43]. As it will be detailed in Section 2.1, the weak separability axiom is the key axiom that justifies the construction of utility functions, that is partial preference relations over the points of view, from the overall preference relation \(\succeq\). A criterion is defined as a preference relation \(\succeq_i\) over one point of view \(X_i\). Thus a criterion is the association of one point of view \(X_i\) with its related utility function \(u_i\).

In practice, we restrict ourself to a family \(\mathcal{F}\) of aggregation functions (parameterized by some coefficients). The justification of the use of a special family is based on an axiomatic approach. The axioms that characterize the family should be in accordance with the problem in consideration and the behaviour of the decision maker. The DM has then to provide the needed information to set the parameters of the model. The more restrictive the family is, the less representative it is, but the less information the DM shall give.

The most classical functions used to aggregate the criteria are the weighted sums \(F(u_1, \ldots, u_n) = \sum_{i=1}^n \alpha_i u_i\). As an aggregation operator, they are characterized by an independence axiom [42, 73]. This property implies some limitations in the way the weighted sum can model typical decision behaviours.