Chapter 11

GENERALIZED CONVEXITY,
GENERALIZED MONOTONICITY
AND NONSMOOTH ANALYSIS

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Abstract This chapter is an introduction to generalized monotone multivalued maps and their relation to generalized convex functions through subdifferential theory. In particular, it contains the characterization of various types of generalized convex functions through properties of their subdifferentials. Also, some recent results on properly quasimonotone maps, maximal pseudomonotone maps, and a new “quasiconvex” subdifferential are presented.

Keywords: Generalized convexity, generalized monotonicity, subdifferential, maximal monotone operators.

1. Introduction

One of the cornerstones of Convex Analysis is the use of the subdifferential of convex functions. The reason is of course that in most problems arising in practice, convex functions are not differentiable, and in this case the subdifferential (also called Fenchel-Moreau subdifferential) provides an elegant and powerful tool for their study. In spite of its importance, the class of convex functions is a very small one. Since the advances of B.N. Pshenichnyi in Nonsmooth Optimization [64] and F. Clarke [14] who extended the notion of subdifferential to the class of locally Lipschitz functions, considerable effort has been devoted to create subdifferentials that can be used in nonconvex problems. As is usual in
cases of generalization, the result was the definition of a large number of subdifferentials, each of which was introduced having in mind a particular application, or a property of the Fenchel-Moreau subdifferential that one wishes to preserve. Of course, none of these subdifferentials is best in all situations.

Given the variety of subdifferentials, it is remarkable that most of them are suitable for characterizing generalized convex functions. In fact, at first one has to introduce appropriate multivalued generalizations of the single valued generalized monotone maps. Having done so, one has then to show that a function is generalized convex in some sense, if and only if its subdifferential is generalized monotone in a corresponding sense. The literature on this subject is vast, and not easy to follow. The complexity of the field is mainly the result of the multitude of generalized monotonicity notions, the large number of subdifferentials, and the different assumptions of continuity etc., used by each author. In an effort to show the underlying unity, several authors proposed axiomatic schemes that cover a large number of subdifferentials and showed that many of the results are valid for all subdifferentials that fall into their axiomatic schemes. Let us mention in this respect the work of Penot [55, 57, 60], Aussel, Corvellec and Lassonde [2, 3] and Thibault and Zagrodny [68].

A large part of this chapter is devoted to an exposition of the main results on the relation of generalized convexity and generalized monotonicity via subdifferential theory. We exclude from our exposition all contributions not directly related to all three of these topics, such as: results involving generalized derivatives rather than subdifferentials (see Chapter 10 of this volume), the application of nonsmooth analytical tools and generalized convexity to duality methods, Mathematical Economics, Hamilton Jacobi-equations etc. (see [58] and Chapter 6 of this volume, and the references therein), the calculus of subdifferentials adapted to Generalized Convexity [62]. Even so, we will have to leave aside some important contributions on the integration of generalized monotone maps [7], the characterization of quasiconvexity via the normal cones [4, 8] etc.

The plan of the chapter is as follows. The next section introduces the various kinds of subdifferentials we are going to use. Section 3 studies the relation of convexity to monotonicity, and some variants. Section 4 is devoted to quasiconvexity and quasimonotonicity, and Section 5 concerns pseudoconvexity and pseudomonotonicity. Section 6 studies the recent concept of proper quasimonotonicity, its relation to variational inequalities, and explores the various connections between the notions of generalized monotonicity. Section 7 presents a recently introduced “quasiconvex” subdifferential and some of its properties. The last sec-