SOME CLASSICAL VIEWS ON THE PARAMETERS OF THE GROTHENDIECK-TEICHMÜLLER GROUP

To John Thompson on the occasion of his 70-th birthday

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Abstract

We present two new formulas concerning behaviors of the standard parameters of the Grothendieck-Teichmüller group \( \tilde{GT} \), and discuss their relationships with classical mathematics. First, considering a non-Galois etale cover of \( \mathbb{P}^1 - \{0, 1, \infty\} \) of degree 4, we present a new type equation satisfied by the Galois image in \( \tilde{GT} \). Second, a certain equation in \( \text{GL}_2(\mathbb{Z}[[\mathbb{Z}^2]]) \) satisfied by every element of \( \tilde{GT} \) is derived as an application of (profinite) free differential calculus.

0. Introduction

The structure of the absolute Galois group \( G_\mathbb{Q} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) is one of the most important subjects to study in number theory and arithmetic geometry. One attractive approach has occurred since the fundamental work of G.V.Belyi [Be] published in 1979, which shows that \( G_\mathbb{Q} \) has a faithful representation in the profinite fundamental group of the projective line minus 3 points. In [Gr], A.Grothendieck predicted that a certain profinite group approximating \( G_\mathbb{Q} \) can be formulated from the tower of profinite Teichmüller modular groupoids starting from the initial stage \( \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) \). Based on this significant philosophy, V.G.Drinfeld [Dr] and Y.Ihara [II] introduced the Grothendieck-Teichmüller group \( \tilde{GT} \) in which \( G_\mathbb{Q} \) sits in a standard way. Unfortunately the fundamental question of whether \( G_\mathbb{Q} = \tilde{GT} \) has remained unsettled yet; however, it is possible to look at various behaviors of the image of \( G_\mathbb{Q} \hookrightarrow \tilde{GT} \) from certain geometric, arithmetic viewpoints (cf. [12], [LS], [NT] etc.).

In my Florida talk, I reported two equations of different nature in the profinite Grothendieck-Teichmüller group \( \tilde{GT} \). The first one (Prop. 1.1 below) is derived from the classical (non-Galois) cover of modular curves \( X_0(3) \to X(1) \)
of degree 4. This holds on the image of $G_Q$ in $\hat{G}T$, and is unknown whether to hold on the total $\hat{G}T$. The second formula (Prop. 3.1, below) is derived from application of the classical Magnus-Gassner formalism in combinatorial group theory. This gives an equation of two-by-two matrices of two formal variables (hence produces infinitely many equations by specializations of variables) which holds on the total $\hat{G}T$.

In this note, we present proofs of these equations and related results with attempting to show several background materials from different contexts of classical mathematics. Still, concerning ultimate estimation of a (possible) gap between $G_Q$ and $\hat{G}T$, perspectives have remained obscure from these investigations.

The organization of the sections is as follows: In Sect.1, we summarize a simple typical method (initiated in [NS]-[NT]) to abstract an equation satisfied by $G_Q$ in $\hat{G}T$ from a certain "doubly 3 point ramified" cover of projective lines, and present Prop. 1.1 as an application. In Sect.2, the same method is examined to apply to the non-compact cases of the list of Singerman’s table [Si]. In Sect.3, changed is our focus to the method of classical Magnus-Gassner representations which yields Prop. 3.1. Finally, in Sect.4, we discuss specializations of Prop. 3.1 and discuss several complementary facts.

We refer to [Il], [Sc], [HS] for basic facts on $\hat{G}T$, and write the standard parameter of $\sigma \in \hat{G}T$ as $(f_\sigma, \lambda_\sigma) \in \hat{F}_2 \times \hat{Z}^\times$, where $\hat{F}_2$ is the profinite free group of rank 2 generated by two non-commutative symbols.

1. Hauptmodul and Thompson series

It is now well known that a certain special type of 3 point ramified cover of the projective line $\mathbb{P}^1$ affords newtype equations satisfied by the image of $G_Q \hookrightarrow \hat{G}T$. The first example was given in [NS] Theorem 2.2 using a certain combination of two double covers of $\mathbb{P}^1$. Then, in [NT], we investigated several other examples appearing in the Legendre-Jacobi covers with Galois group $S_3$ (and its subcovers). This cover is essentially the same as the cover $X(2) \to X'(1) = \mathbb{P}^1_j$ of the elliptic modular curve of level 2 over that of level 1 — the $J$-line. In loc.cit., we also introduced the intermediate covers by the harmonic line $\mathbb{P}^1_u = X_0(2)$ and the equianharmonic line $\mathbb{P}^1_v$ of degrees 3, 2 over $X(1)$ respectively, and studied the Galois covers $X(2) \to \mathbb{P}^1_u, X(2) \to \mathbb{P}^1_v$ of degrees 2, 3 respectively.

One finds that the common geometric features of these (Galois) covers $Y(\mathbb{C}) \to X(\mathbb{C})$ are:

(1) Ramification occurs only over the three points $0, 1, \infty$ of $X(\mathbb{C}) = \mathbb{P}^1$. 