PARTITION IDENTITIES FOR THE MULTIPLE ZETA FUNCTION

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Abstract We define a class of expressions for the multiple zeta function, and show how to determine whether an expression in the class vanishes identically. The class of such identities, which we call partition identities, is shown to coincide with the class of identities that can be derived as a consequence of the stuffle multiplication rule for multiple zeta values.

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1. Introduction

For positive integer $n$ and real $s_j \geq 1$ ($j = 1, 2, \ldots, n$) the multiple zeta function may be defined by

$$\zeta(s_1, s_2, \ldots, s_n) = \sum_{k_1 > k_2 > \ldots > k_n > 0} \prod_{j=1}^{n} k_j^{-s_j}. \quad (1.1)$$

The nested sum (1.1) is over all positive integers $k_1, \ldots, k_n$ satisfying the indicated inequalities, and is finite if and only if $s_1 > 1$ also holds. An elementary property of the multiple zeta function is that it satisfies the so-called stuffle multiplication rule [1]: If $\vec{u} = (u_1, \ldots, u_m)$ and $\vec{v} = (v_1, \ldots, v_n)$, then

$$\zeta(\vec{u})\zeta(\vec{v}) = \sum_{\vec{w} \in \vec{u} \ast \vec{v}} \zeta(\vec{w}), \quad (1.2)$$

where $\vec{u} \ast \vec{v}$ is the multi-set of size [2]

$$|\vec{u} \ast \vec{v}| = \sum_{k=0}^{\min(m,n)} \binom{m}{k} \binom{n}{k} 2^k$$
defined by the recursion
\[(s, \tilde{u}) \ast (t, \tilde{v}) = \{(s, \tilde{w}) : \tilde{w} \in \tilde{u} \ast (t, \tilde{v})\} \cup \{(t, \tilde{w}) : \tilde{w} \in (s, \tilde{u}) \ast \tilde{v}\} \]
\[\cup \{(s + t, \tilde{w}) : \tilde{w} \in \tilde{u} \ast \tilde{v}\},\]
with initial conditions \(\tilde{u} \ast () = () \ast \tilde{u} = \tilde{u}\). Thus, for example,
\[(s, u) \ast (t, v) = \{(s, u, t, v), (s, u + t, v), (s, t, u, v), (s, t, u + v), (s, t, v, u)\} \]
\[\cup \{(t, s, u, v), (t, s, u + v), (t, s, v, u), (t, s + v, u), (t, v, s, u)\} \]
\[\cup \{(s + t, u, v), (s + t, u + v), (s + t, v, u)\},\]
and correspondingly, we have the stuffle identity
\[\zeta(s, u)\zeta(t, v)\]
\[= \zeta(s, u, t, v) + \zeta(s, u + t, v) + \zeta(s, t, u, v) + \zeta(s, t, u + v) + \zeta(s, t, v, u) + \zeta(t, s, u, v) + \zeta(t, s, u + v) + \zeta(t, s, v, u) + \zeta(t, s + v, u) + \zeta(t, v, s, u) + \zeta(s + t, u, v) + \zeta(s + t, u + v) + \zeta(s + t, v, u).\]
The sum on the right hand side of equation (1.2) accounts for all possible interlacings of the summation indices when the two nested series on the left are multiplied.

In this paper, we consider a certain class of expressions (“legal expressions”) for the multiple zeta function, consisting of a finite linear combination of terms. Roughly speaking, a term is a product of multiple zeta functions, each of which is evaluated at a sequence of sums selected from a common argument list \((s_1, \ldots, s_n)\) in such a way that each variable \(s_j\) appears exactly once in each term. A more precise definition is given in Section 2. Once the legal expressions have been defined, we consider the problem of determining when a legal expression vanishes identically. For reasons which will become clear, we call such identities partition identities. It will be seen that the problem of verifying or refuting an alleged partition identity reduces to finite arithmetic over a polynomial ring. Alternatively, one can first rewrite any legal expression as a sum of single multiple zeta functions by applying the stuffle multiplication rule to each term. As we shall see, it is then easy to determine whether or not the original expression vanishes identically.

2. Definitions

Our definition of a partition identity makes use of the concept of a set partition. It is helpful to distinguish between set partitions that are ordered and those that are unordered.

Definition 1 (Unordered Set Partition). Let \(S\) be a finite non-empty set. An unordered set partition of \(S\) is a finite non-empty set