Chapter 19

THE DIRECT METHOD FOR A CLASS OF INFINITE HORIZON DYNAMIC GAMES

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Abstract In this paper we present an extension of a direct solution method, originally due to Leitmann (1967) for single-player games on a finite time interval, to a class of infinite horizon N-player games in which the state equation is affine in the strategies of the players. Our method, based on a coordinate transformation method, gives sufficient conditions for an open-loop Nash equilibrium. An example is presented to illustrate the utility of our results.

1. Introduction

Recently, there has been a series of papers by the authors in which a direct solution method has been used to obtain solutions to open-loop finite-horizon differential games with prescribed two-point boundary conditions. The purpose of this paper is to extend the direct method, alluded to above, to address problems defined on the infinite time horizon. Problems of this type have many important applications in economics and consequently the extension of the direct method to these types of models will significantly enlarge the class of applied problems to which this method can be utilized.

C. Deissenberg and R.F. Hartl (eds.), Optimal Control and Dynamic Games, 319–334.

2. The general model considered

We consider an N-player game in which the dynamics of the j-th player, $j = 1, 2, \ldots, N$, at time $t \geq t_0$ is denoted by $\mathbf{x}_j(t)$, generated by control $u_j(t)$,and satisfies an affine control system of the form

$$\dot{\mathbf{x}}_j(t) = F_j(t, \mathbf{x}(t)) + G_j(t, \mathbf{x}(t))u_j(t), \quad \text{a.e.} \quad t_0 \leq t \tag{19.1}$$

with fixed initial condition,

$$\mathbf{x}_j(t_0) = \mathbf{x}_{0j}, \tag{19.2}$$

control constraints

$$u_j(t) \in U_j(t) \subset \mathbb{R}^{m_j}, \quad \text{a.e.} \quad t_0 \leq t \tag{19.3}$$

and state constraints

$$\mathbf{x}_j(t) \in X_j(t) \subset \mathbb{R}^{n_j}, \quad \text{for} \quad t_0 \leq t \tag{19.4}$$

We assume here that for each $j = 1, 2, \ldots, N$ $\mathbf{x}_j(\cdot) : [t_0, +\infty) \to \mathbb{R}^{n_j}$, $u_j(\cdot) : [t_0, +\infty) \to \mathbb{R}^{m_j}$, and $\mathbf{x}(\cdot) \doteq (\mathbf{x}_1(\cdot), \mathbf{x}_2(\cdot), \ldots, \mathbf{x}_N(\cdot)) : [t_0, +\infty) \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_N} = \mathbb{R}^{\mathbf{n}}$. The functions $F_j(\cdot, \cdot) : [t_0, +\infty) \times \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{n_j}$ and $G_j(\cdot, \cdot) : [t_0, +\infty) \times \mathbb{R}^{\mathbf{n}} \to \mathbb{R}^{n_j \times m_j}$ are continuous for each $j = 1, 2, \ldots, N$ and also for each $t \in [t_0, +\infty)$ sets $X_j(t)$ and $U_j(t)$ are convex subsets of $\mathbb{R}^{n_j}$ and $\mathbb{R}^{m_j}$. Additionally, we assume that each matrix, $G_j(t, \mathbf{x})$, for $(t, \mathbf{x}) \in [t_0, +\infty) \times \mathbb{R}^{\mathbf{n}}$, has a left inverse $G_j(t, \mathbf{x})^{-1}$ that is also continuous.

The objective of each player is to minimize a performance criterion of the form,

$$J_j(\mathbf{x}(\cdot), u_j(\cdot)) = \int_{t_0}^{+\infty} F_j^0(t, \mathbf{x}(t), u_j(t))\, dt \tag{19.5}$$

in which $F_j^0(\cdot, \cdot, \cdot) : [t_0, +\infty) \times \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{m_j} \to \mathbb{R}$ is assumed to be continuous. Clearly, it is unreasonable to expect that each player will be able to minimize his/her performance and consequently we seek a Nash equilibrium. To define this we have the following definitions and notation.

Definition 1 We say a pair of functions $\{\mathbf{x}(\cdot), \mathbf{u}(\cdot)\} : [t_0, +\infty) \to \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{m}}$ is an admissible trajectory-strategy pair if and only if $t \to \mathbf{x}(t)$ is locally absolutely continuous on $[t_0, +\infty)$ (i.e., it is absolutely continuous on each compact subinterval of $[t_0, +\infty)$), $t \to \mathbf{u}(t)$ is Lebesgue measurable on $[t_0, +\infty)$, for each $j = 1, 2, \ldots, N$, the relations (19.1)–(19.4) are satisfied, and for each $j = 1, 2, \ldots, N$, the functionals (19.5) are finite Lebesgue integrals.