Chapter 12

COMPUTATIONAL SPATIAL AND MATERIAL SETTINGS OF CONTINUUM MECHANICS. AN ARBITRARY LAGRANGIAN EULERIAN FORMULATION

Ellen Kuhl
Chair of Applied Mechanics, Faculty of Mechanical Engineering
University of Kaiserslautern, Germany
ekuhl@rhrk.uni-kl.de

Harm Askes
Computational Mechanics, Faculty of Civil Engineering and Geosciences
Delft University of Technology, Netherlands
h.askes@citg.tudelft.nl

Paul Steinmann
Chair of Applied Mechanics, Faculty of Mechanical Engineering
University of Kaiserslautern, Germany
ps@rhrk.uni-kl.de

Abstract

The present contribution aims at deriving a generic hyperelastic Arbitrary Lagrangian Eulerian formulation embedded in a consistent variational framework. The governing equations follow straightforwardly from the Dirichlet principle for conservative mechanical systems. Thereby, the key idea is the reformulation of the total variation of the potential energy at fixed referential coordinates in terms of its variation at fixed material and at fixed spatial coordinates. The corresponding Euler–Lagrange equations define the spatial and the material motion version of the balance of linear momentum, i.e. the balance of spatial and material forces, in a consistent dual format. In the discretised setting, the governing equations are solved simultaneously rendering
the spatial and the material configuration which minimise the overall potential energy of the system. The remeshing strategy of the ALE formulation is thus no longer user-defined but objective in the sense of energy minimisation. As the governing equations are derived from a potential, they are inherently symmetric, both in the continuous case and in the discrete case.

**Keywords:** Arbitrary Lagrangian Eulerian formulation, spatial and material settings, variational principle, spatial and material forces, finite element technology.

**Introduction**

The deformation of a body is essentially characterised through the balance of linear momentum. Typically, this balance of momentum is formulated in terms of the spatial deformation map mapping the material placements of particles in the material configuration to their spatial placements in the spatial configuration. Alternatively, we could formulate the balance of momentum in terms of the material motion map, see [Maugin, 1993; Gurtin, 2000; Kienzler and Herrmann, 2000; Steinmann, 2002a; Steinmann, 2002b; Kuhl and Steinmann, 2003]. Being the inverse of the spatial motion map, the material motion map characterises the mapping of particles in the spatial configuration to their material motion counterparts. While the former approach relates to the equilibration of the classical spatial forces in the sense of Newton, the latter lends itself to the equilibration of material forces in the sense of Eshelby.

The finite element discretisation renders the discrete residual statement of the balance of momentum. In the classical sense, the solution of the finite element method corresponds to the vanishing residual of the spatial force balance which is related to the minimum of the potential energy. This minimum, however, is only a minimum with respect to fixed material placements.

In the discrete setting, a vanishing residual of discrete spatial forces does not necessarily imply that the discrete material forces vanish equivalently, see [Braun, 1997; Mueller and Maugin, 2002; Kuhl et al., 2003; Askes et al., 2003; Thoutireddy, 2003]. Non-vanishing discrete material forces indicate that the underlying discretisation is not yet optimal. An additional release of energy can take place when finite element nodes are moved in the direction opposite to the resulting discrete material node point forces. In this respect, the proposed strategy can be interpreted as a particular version of an Arbitrary Lagrangian Eulerian formulation, see e.g. [Donea, 1980; Hughes et al., 1981] for ALE formulations in gen-