4

Distributions

In many problems such as the analysis of data from radioactive decay, the distribution of observations can be determined on theoretical grounds, and the optimal decision procedure is one that takes advantage of knowledge obtained this way. In this chapter, we consider optimal tests for data drawn from the binomial, Poisson, exponential, uniform, and exponential family of distributions.

4.1 Properties of Independent Observations

All the tests that we have considered so far require that the individual observations be independent of one another. We will continue to make such an assumption in succeeding chapters, even in cases when the "observation" consists of a vector of interdependent variables. This is because independent observations have many desirable properties. Recall that if X and Y are independent, then $\Pr\{X \in A \text{ and } Y \in B\} = \Pr\{X \in A\}\Pr\{Y \in B\}$. In consequence, if $E|X| < \infty$ and $E|Y| < \infty$, we easily can show that $E(aX+b) = aEX+b$ where a and b are constants, and $E(X+Y) = E(X)+E(Y)$ (Exercise 4.1). If Var $X < \infty$, and Var $Y < \infty$, then Var $(aX+b) = a^2$ Var X, Var $(X+Y) =$ Var $X +$ Var Y, and Var $(X-Y) =$ Var $X +$ Var Y. We take advantage of all these properties in what follows.

In particular, because Var $(aX+b) = a^2$ Var X, the mean of 100 independent identically distributed observations has 1/100 the variance of a single observation; the standard deviation of the mean of 100 observations, termed the *standard error*, has 1/10 the standard deviation of a single observation. We'll use this latter property of the mean in Sections 13.1.3 and 14.10 when we try to determine how large a sample should be.

4.2 Binomial Distribution

A binomial frequency distribution, written $B(n,p)$, results from a series of n independent trials each with the same probability p of success, and the

same probability $q = 1 - p$ of failure. This distribution arises in any context in which there are only two possible outcomes, such as "the patient recovers or the patient dies." The distribution that arises when we can prefer Coke to Pepsi, Pepsi to Coke, have no preference, or prefer some other beverage entirely, is termed a *multinomial* and will be considered in Chapter 8.

If X is $B(n, p)$, then the expected value of X denoted by EX is np and the variance of X about its expected value, $EX - E(X)^2$, denoted by Var X, is npq.

Suppose we've flipped a coin in the air seven times, and six times it has come down heads. Do we have reason to suspect the coin is not fair, that $p > 1/2$?

To answer this question, we need to look at the frequency distribution of the binomial observation. If X is $B(n, p)$, then $\Pr\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = 0, 1, \ldots, n$, and is zero otherwise. Note that we needn't keep track of the individual observations; the number of successes k is *sufficient* for testing hypotheses concerning p.

If $n = 7$ and $k = 6$, this probability is $7p^6(1 - p)$. If $p = 1/2$, this probability is $7/128 = 0.0547$, just slightly more than 5%. But before we reject the hypothesis that ours is a fair coin, consider that if six heads out of seven tries seems extreme to us, seven heads out of seven would seem even more extreme. Adding the probability of this more extreme event to what we have already, we see the probability of throwing six or more heads in seven tries is $8/128 = 0.0625$. While not significant at the 5% level, six or more heads out of seven tries does seem suspicious. If you were a mad scientist and observed that six times out of seven your assistant Igor began to bay like a wolf when there was a full moon, wouldn't you get suspicious?

Warning: In the example of the unfair coin, we formulated our hypothesis *after* we observed the data. While we might have reason to be suspicious and enough reason to justify further testing, we would be in error if we reported a significance level of 0.0625. When we are observing more or less at random, without a specific hypothesis in mind, human beings being what they are, the probability that sooner or later we will see something interesting is one. (Next time you play pool, try naming both the ball and the pocket.)

4.3 Poisson: Events Rare in Time and Space

The decay of a radioactive element, an appointment to the United States Supreme Court and a cavalry officer trampled by his horse have in common that they are relatively rare but inevitable events. They are inevitable, that is, if there are enough atoms, enough seconds or years in the observation period, and enough horses and momentarily careless men. Their frequency of occurrence has a Poisson distribution.