Ramanujan’s Continued Fraction for
\((q^2; q^3)_\infty / (q; q^3)_\infty\)

8.1 Introduction

In Chapter 6, we proved some general theorems on continued fractions from the lost notebook that yielded several beautiful examples as special cases, in particular, the Rogers–Ramanujan continued fraction, the Ramanujan–Göllnitz–Gordon continued fraction, and Ramanujan’s cubic continued fraction. In Chapter 7, we considered asymptotic formulas for continued fractions, but one of the examples on which we focused in that chapter does not fall under the purview of the general theorems in Chapter 6. Our goal in this chapter is to prove two remarkable theorems for this continued fraction

\[
(q^2; q^3)_\infty / (q; q^3)_\infty = \frac{1}{1 - q + q^2 - 1 + q^2 - 1 + q^3 - \cdots}, \quad |q| < 1.
\] (8.1.1)

The continued fraction (8.1.1) is due to Ramanujan and is found in his second notebook [227, p. 290]. Of the many \(q\)-continued fractions found by Ramanujan, (8.1.1) is, by far, the most difficult to prove. Up until recently, the only known proof was found by Andrews, Berndt, L. Jacobsen, and R.L. Lamphere [39], [63, p. 46, Entry 19] in 1992, which uses a deep theorem of Andrews [17]. However, a considerably shorter and more natural proof was recently given by Andrews, Berndt, J. Sohn, A.J. Yee, and A. Zaharescu [40].

On page 45 in his lost notebook, Ramanujan claims, in an unorthodox fashion, that a certain \(q\)-continued fraction possesses three limit points. More precisely, he asserts that as \(n\) tends to \(\infty\) in the three residue classes modulo 3, the \(n\)th partial quotients tend, respectively, to three distinct limits, which he explicitly gives. In fact, Ramanujan claims that a more general continued fraction has three distinct limits under the broader concept of “general convergence,” which was not defined in the literature until about 70 years later. If \(\omega = e^{2\pi i/3}\), then, except for the simplification of notation, Ramanujan [228, p. 45] claimed that for \(|q| < 1\),
\[
\lim_{n \to \infty} \left( \frac{1}{1 - \frac{1}{1 + q - 1 + q^2 - \cdots - 1 + q^n + a}} \right)
= -\omega^2 \left( \frac{\Omega - \omega^{n+1}}{\Omega - \omega^n - 1} \right), \quad (8.1.2)
\]
where
\[
\Omega := \frac{1 - a\omega^2}{1 - a\omega} \frac{\omega^2q}{(\omega q; q)_\infty}. \quad (8.1.3)
\]

After (8.1.2), Ramanujan appended the note, “Numerator and Denominators can be equated separately.”

Of course, because of the appearance of the limiting variable \( n \) on the right side of (8.1.2), Ramanujan’s claim is meaningless as it stands. But after a few minutes of reflection, we readily conclude that Ramanujan was affirming that there are three distinct limits depending on the congruence class modulo 3 in which \( n \to \infty \). In the note after (8.1.3), Ramanujan evidently asserted that the limits can be obtained by determining separately the limits of both the partial numerators and denominators.

Ramanujan’s claim is very interesting for several reasons.

First, if \( a = 0 \), the left side of (8.1.2) is a continued fraction (in the normal sense) that diverges. We prove that the three partial quotients tend to the required limits if \( n \) is restricted to any one of the three residue classes modulo 3. This is in contrast to the classical result from the general theory of continued fractions, which asserts that if all the elements of a divergent continued fraction are positive, then the even and odd approximants approach distinct limits [182, pp. 96–97].

Second, if \( a \neq 0 \), we prove that the continued fraction in (8.1.2) converges “generally” in the sense that when \( n \) is confined to any one of the three residue classes modulo 3, the limit of the left side indeed exists and is equal to that claimed on the right side of (8.1.2) in each of the three cases. The concept of general convergence is due to L. Jacobsen [167] in 1986. See also her book with H. Waadeland [182, pp. 41–44]. For some results of Ramanujan of a different kind on general convergence, see Chapter 5. Thus, we have one further example of Ramanujan’s having discovered a fundamental concept long ahead of his time, before anyone else ever thought of it.

Third, note that the continued fraction (8.1.1) can be written in the equivalent form
\[
\frac{(q^2; q^3)_\infty}{(q; q^3)_\infty} = \frac{1}{1 - \frac{1}{q^{-1} + 1} - \frac{1}{q^{-2} + 1} - \frac{1}{q^{-3} + 1} - \cdots. \quad (8.1.4)
\]

Thus, when \( a = 0 \), the continued fraction on the left side of (8.1.2) is the same as the continued fraction of (8.1.4), but with \( q \) replaced by \( 1/q \). Observe that, remarkably, \( (q^2; q^3)_\infty/(q; q^3)_\infty \) also appears in the three limits on the right side of (8.1.2). In this sense, Ramanujan’s result (8.1.2) is analogous to his theorem on the divergence of the Rogers–Ramanujan continued fraction found