1
Basic Theory

1.1 Points and Vectors

Real life methods for constructing curves and surfaces often start with points and vectors, which is why we start with a short discussion of the properties of these mathematical entities. The material in this section applies to both two-dimensional and three-dimensional points and vectors, while the examples are given in two-dimensions.

Points and vectors are different mathematical entities. A point has no dimensions; it represents a location in space. A vector, on the other hand, has no well-defined location and its only attributes are direction and magnitude. People tend to confuse points and vectors because it is natural to associate a point $\mathbf{P}$ with the vector $\mathbf{v}$ that points from the origin to $\mathbf{P}$ (Figure 1.1a). This association is useful, but the reader should bear in mind that $\mathbf{P}$ and $\mathbf{v}$ are different.

Both points and vectors are represented by pairs or triplets of real numbers, but these numbers have different meanings. A point with coordinates $(3, 4)$ is located 3 units to the right of the y axis and 4 units above the x axis. A vector with components $(3, 4)$, however, points in direction $4/3$ (it moves 3 units in the x direction for every 4 units in the y direction, so its slope is $4/3$) and its magnitude is $\sqrt{3^2 + 4^2} = 5$. It can be located anywhere.

In mathematics, entities are always associated with operations. An entity that cannot be operated on is generally not useful. Thus, we discuss operations on points and vectors. The first operation is to multiply a point $\mathbf{P}$ by a real number α. The product $\alpha\mathbf{P}$ is a point on the line connecting $\mathbf{P}$ to the origin (Figure 1.1b). Note that this line is infinite and $\alpha\mathbf{P}$ can be located anywhere on it, depending on the value of α.

The next operation is subtracting points. Let $\mathbf{P}_0 = (x_0, y_0)$ and $\mathbf{P}_1 = (x_1, y_1)$ be two points. The difference $\mathbf{P}_1 - \mathbf{P}_0 = (x_1 - x_0, y_1 - y_0) = (\Delta x, \Delta y)$ is well defined. It is the vector (the direction and distance) from $\mathbf{P}_0$ to $\mathbf{P}_1$ (Figure 1.1b).

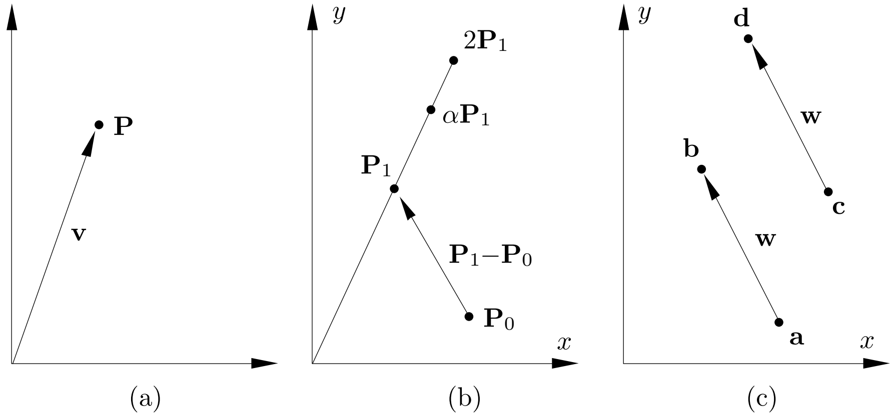

Figure 1.1: Operations on Points.

Figure 1.1c shows two pairs of points **a b** and **c d**. Points **a** and **c** are different and so are **b** and **d**. The vectors $\mathbf{b}-\mathbf{a}$ and $\mathbf{d}-\mathbf{c}$, however, are identical

Example: The two points $\mathbf{P}_0 = (5,4)$ and $\mathbf{P}_1 = (2,6)$ are subtracted to produce the pair $\mathbf{P}_1 - \mathbf{P}_0 = (-3,2)$. The new pair is a vector, because it represents a direction and a distance. To get from $\mathbf{P}_0$ to $\mathbf{P}_1$, we need to move -3 units in the x direction and 2 units in the y direction. Similarly, $\mathbf{P}_0 - \mathbf{P}_1$ is the direction from $\mathbf{P}_1$ to $\mathbf{P}_0$. The distance between the points is $\sqrt{(-3)^2+2^2}$. These properties do not depend on the particular coordinate axes used. If we translate the origin—or, equivalently, translate the points—m units in the x direction and n units in the y direction, the points will have new coordinates, but the difference will not change. The same property (the difference of points being independent of the coordinate axes) holds after rotation, scaling, shearing, and reflection: the so-called *affine transformations* (or mappings). This is why the operation of subtracting two points is affinely invariant. (Note that the product $\alpha\mathbf{P}$ is also affinely invariant.)

The sum of a point and a vector is well defined and is a point. Figure 1.2a shows the two sums $\mathbf{P}_1^* = \mathbf{P}_1 + \mathbf{v}$ and $\mathbf{P}_2^* = \mathbf{P}_2 + \mathbf{v}$. It is easy to see that the relative positions of $\mathbf{P}_1^*$ and $\mathbf{P}_2^*$ are the same as those of $\mathbf{P}_1$ and $\mathbf{P}_2$. Another way to look at the sum $\mathbf{P}+\mathbf{v}$ is to observe that it moves us away from $\mathbf{P}$, which is a point, in a certain direction and by a certain distance, thereby bringing us to another point. Yet another way of showing the same thing is to rewrite the relation $\mathbf{a}-\mathbf{b}=\mathbf{v}$ as $\mathbf{a}=\mathbf{b}+\mathbf{v}$, which shows that the sum of point $\mathbf{b}$ and vector $\mathbf{v}$ is a point $\mathbf{a}$.

Given any two points $\mathbf{P}_0$ and $\mathbf{P}_2$, the expression $\mathbf{P}_0 + \alpha(\mathbf{P}_2 - \mathbf{P}_0)$ is the sum of a point and a vector, so it is a point that we can denote by $\mathbf{P}_1$. The vector $\mathbf{P}_2 - \mathbf{P}_0$ points from $\mathbf{P}_0$ to $\mathbf{P}_2$, so adding it to $\mathbf{P}_0$ produces a point on the line connecting $\mathbf{P}_0$ to $\mathbf{P}_2$. Thus, we conclude that the three points $\mathbf{P}_0$, $\mathbf{P}_1$, and $\mathbf{P}_2$ are collinear. Note that the expression $\mathbf{P}_1 = \mathbf{P}_0 + \alpha(\mathbf{P}_2 - \mathbf{P}_0)$ can be written $\mathbf{P}_1 = (1-\alpha)\mathbf{P}_0 + \alpha\mathbf{P}_2$, showing that $\mathbf{P}_1$ is a linear combination of $\mathbf{P}_0$ and $\mathbf{P}_2$. In general, any of three collinear points can be written as a linear combination of the other two. Such points are not independent.

◇ **Exercise 1.1:** Given the three points $\mathbf{P}_0 = (1,1)$, $\mathbf{P}_1 = (2,2.5)$, and $\mathbf{P}_2 = (3,4)$, are they collinear?