2
Linear Interpolation

In order to achieve realism, the many algorithms and techniques employed in computer graphics have to construct mathematical models of curved surfaces, models that are based on curves. It seems that straight line segments and flat surface patches, which are simple geometric figures, cannot play an important role in achieving realism, yet they turn out to be useful in many instances. A smooth curve can be approximated by a set of short straight segments. A smooth, curved surface can similarly be approximated by a set of surface patches, each a small, flat polygon. Thus, this chapter discusses straight lines and flat surfaces that are defined by points. The application of these simple geometric figures to computer graphics is referred to as *linear interpolation*. The chapter also presents two types of surfaces, bilinear and lofted, that are curved, but are partly based on straight lines.

2.1 Straight Segments

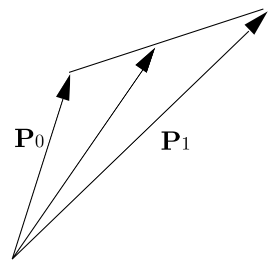
We start with the parametric equation of a straight segment. Given any two points $\mathbf{A}$ and $\mathbf{C}$, the expression $\mathbf{A} + \alpha(\mathbf{C} - \mathbf{A})$ is the sum of a point and a vector, so it is a point (see page 2) that we can denote by $\mathbf{B}$. The vector $\mathbf{C} - \mathbf{A}$ points from $\mathbf{A}$ to $\mathbf{C}$, so adding it to $\mathbf{A}$ results in a point on the line connecting $\mathbf{A}$ to $\mathbf{C}$. Thus, we conclude that the three points $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are collinear. Note that the expression $\mathbf{B} = \mathbf{A} + \alpha(\mathbf{C} - \mathbf{A})$ can be written $\mathbf{B} = (1 - \alpha)\mathbf{A} + \alpha\mathbf{C}$, showing that $\mathbf{B}$ is a linear combination of $\mathbf{A}$ and $\mathbf{C}$ with barycentric weights. In general, any of three collinear points can be written as a linear combination of the other two. Such points are not independent.

We therefore conclude that given two arbitrary points $\mathbf{P}_0$ and $\mathbf{P}_1$, the parametric representation of the line segment from $\mathbf{P}_0$ to $\mathbf{P}_1$ is

$$\mathbf{P}(t) = (1 - t)\mathbf{P}_0 + t\mathbf{P}_1 = \mathbf{P}_0 + (\mathbf{P}_1 - \mathbf{P}_0)t = \mathbf{P}_0 + t\mathbf{d}, \quad \text{for} \quad 0 \le t \le 1. \tag{2.1}$$

The tangent vector of this line is the constant vector $\frac{d\mathbf{P}(t)}{dt} = \mathbf{P}_1 - \mathbf{P}_0 = \mathbf{d}$, the direction from $\mathbf{P}_0$ to $\mathbf{P}_1$.

If we think of $\mathbf{P}_i$ as the vector from the origin to point $\mathbf{P}_i$, then the figure on the right shows how the straight line is obtained as a linear, barycentric combination of the two vectors $\mathbf{P}_0$ and $\mathbf{P}_1$, with coefficients $(1-t)$ and t. We can think of this combination as a vector that pivots from $\mathbf{P}_0$ to $\mathbf{P}_1$ while varying its magnitude, so its tip always stays on the line.



The expression $\mathbf{P}_0 + t\mathbf{d}$ is also useful. It describes the line as the sum of the point $\mathbf{P}_0$ and the vector $t\mathbf{d}$, a vector pointing from $\mathbf{P}_0$ to $\mathbf{P}_1$, whose magnitude depends on t. This representation is useful in cases where the direction of the line and one point on it are known. Notice that varying t in the interval $[-\infty, +\infty]$ constructs the infinite line that contains $\mathbf{P}_0$ and $\mathbf{P}_1$.

2.1.1 Distance of a Point From a Line

Given a line in parametric form $\mathbf{L}(t) = \mathbf{P}_0 + t\mathbf{v}$ (where $\mathbf{v}$ is a vector in the direction of the line) and a point $\mathbf{P}$, what is the distance between them? Assume that $\mathbf{Q}$ is the point on $\mathbf{L}(t)$ that's the closest to $\mathbf{P}$. Point $\mathbf{Q}$ can be expressed as $\mathbf{Q} = \mathbf{L}(t_0) = \mathbf{P}_0 + t_0\mathbf{v}$ for some t_0. The vector from $\mathbf{Q}$ to $\mathbf{P}$ is $\mathbf{P}-\mathbf{Q}$. Since $\mathbf{Q}$ is the nearest point to $\mathbf{P}$, this vector should be perpendicular to the line. Thus, we end up with the condition $(\mathbf{P}-\mathbf{Q})\bullet\mathbf{v} = 0$ or $(\mathbf{P} - \mathbf{P}_0 - t_0\mathbf{v})\bullet\mathbf{v} = 0$, which is satisfied by

$$t_0 = \frac{(\mathbf{P} - \mathbf{P}_0)\bullet\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}}.$$

Substituting this value of t_0 in the line equation gives

$$\mathbf{Q} = \mathbf{P}_0 + \frac{(\mathbf{P} - \mathbf{P}_0)\bullet\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}}\mathbf{v}. \tag{2.2}$$

The distance between $\mathbf{Q}$ and $\mathbf{P}$ is the magnitude of vector $\mathbf{P}-\mathbf{Q}$.

This method always works since vector $\mathbf{v}$ cannot be zero (otherwise there would be no line).

In the two-dimensional case, the line can be represented explicitly as $y = ax+b$ and the problem can be easily solved with just elementary trigonometry. Figure 2.1 shows a general point $\mathbf{P} = (P_x, P_y)$ at a distance d from a line $y = ax + b$. It is easy to see that the vertical distance e between the line and $\mathbf{P}$ is $|P_y - aP_x - b|$. We also know from trigonometry that

$$1 = \sin^2\alpha + \cos^2\alpha = \tan^2\alpha\cos^2\alpha + \cos^2\alpha = \cos^2\alpha(1+\tan^2\alpha),$$

implying

$$\cos^2\alpha = \frac{1}{1+\tan^2\alpha}.$$