3 Polynomial Interpolation

Definition: A polynomial of degree n in x is the function

$$P_n(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where a_i are the coefficients of the polynomial (in our case, they are real numbers). Note that there are $n+1$ coefficients.

Calculating a polynomial involves additions, multiplications, and exponentiations, but there are two methods that greatly simplify this calculation. They are the following:

1. Horner's rule. A degree-3 polynomial can be written in the form

$$P(x) = \big((a_3 x + a_2)x + a_1\big)x + a_0,$$

thereby eliminating all exponentiations.

2. Forward differences. This is one of Newton's many contributions to mathematics and it is described in some detail in Section 1.5.1. Only the first step requires multiplications. All other steps are performed with additions and assignments only.

Given a set of points, it is possible to construct a polynomial that when plotted passes through the points. When fully computed and displayed, such a polynomial becomes a curve that's referred to as a *polynomial interpolation* of the points. The first part of this chapter discusses methods for polynomial interpolation and shows their limitations. The second part extends the discussion to a two-dimensional grid of points, and shows how to compute a two-parameter polynomial that passes through the points. When fully computed and displayed, such a polynomial becomes a surface. The methods described here apply the algebra of polynomials to the geometry of curves and surfaces, but this application is limited, because high-degree polynomials tend to oscillate. Section 1.5, and especially Exercise 1.20 show why this is so. Still, there are cases where high-degree polynomials are useful.

This chapter starts with a simple example where four points are given and a cubic polynomial that passes through them is derived from first principles. Following this, the Lagrange and Newton polynomial interpolation methods are introduced. The chapter continues with a description of several simple surface algorithms based on polynomials. It concludes with the Coons and Gordon surfaces, which also employ polynomials.

3.1 Four Points

Four points (two-dimensional or three-dimensional) $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ are given. We are looking for a PC curve that passes through these points and has the form

$$\mathbf{P}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d} = (t^3, t^2, t, 1)(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})^T = \mathbf{T}(t)\mathbf{A} \quad \text{for} \quad 0 \le t \le 1, \tag{3.1}$$

where each of the four coefficients $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ is a pair (or a triplet), $\mathbf{T}(t)$ is the row vector $(t^3, t^2, t, 1)$, and $\mathbf{A}$ is the column vector $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})^T$. The only unknowns are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$.

Since the four points can be located anywhere, we cannot assume anything about their positions and we make the general assumption that $\mathbf{P}_1$ and $\mathbf{P}_4$ are the two endpoints $\mathbf{P}(0)$ and $\mathbf{P}(1)$ of the curve, and that $\mathbf{P}_2$ and $\mathbf{P}_3$ are the two interior points $\mathbf{P}(1/3)$ and $\mathbf{P}(2/3)$. (Having no information about the locations of the points, the best we can do is to use equi-distant values of the parameter t.) We therefore write the four equations $\mathbf{P}(0) = \mathbf{P}_1$, $\mathbf{P}(1/3) = \mathbf{P}_2$, $\mathbf{P}(2/3) = \mathbf{P}_3$, and $\mathbf{P}(1) = \mathbf{P}_4$, or explicitly

$$\begin{aligned}
\mathbf{a}(0)^3 + \mathbf{b}(0)^2 + \mathbf{c}(0) + \mathbf{d} &= \mathbf{P}_1,\\
\mathbf{a}(1/3)^3 + \mathbf{b}(1/3)^2 + \mathbf{c}(1/3) + \mathbf{d} &= \mathbf{P}_2,\\
\mathbf{a}(2/3)^3 + \mathbf{b}(2/3)^2 + \mathbf{c}(2/3) + \mathbf{d} &= \mathbf{P}_3,\\
\mathbf{a}(1)^3 + \mathbf{b}(1)^2 + \mathbf{c}(1) + \mathbf{d} &= \mathbf{P}_4.
\end{aligned} \tag{3.2}$$

The solutions of this system of equations are

$$\begin{aligned}
\mathbf{a} &= -(9/2)\mathbf{P}_1 + (27/2)\mathbf{P}_2 - (27/2)\mathbf{P}_3 + (9/2)\mathbf{P}_4,\\
\mathbf{b} &= 9\mathbf{P}_1 - (45/2)\mathbf{P}_2 + 18\mathbf{P}_3 - (9/2)\mathbf{P}_4,\\
\mathbf{c} &= -(11/2)\mathbf{P}_1 + 9\mathbf{P}_2 - (9/2)\mathbf{P}_3 + \mathbf{P}_4,\\
\mathbf{d} &= \mathbf{P}_1.
\end{aligned} \tag{3.3}$$

Substituting these solutions into Equation (3.1) gives

$$\begin{aligned}
\mathbf{P}(t) =& \big(-(9/2)\mathbf{P}_1 + (27/2)\mathbf{P}_2 - (27/2)\mathbf{P}_3 + (9/2)\mathbf{P}_4\big)t^3\\
&+ \big(9\mathbf{P}_1 - (45/2)\mathbf{P}_2 + 18\mathbf{P}_3 - (9/2)\mathbf{P}_4\big)t^2\\
&+ \big(-(11/2)\mathbf{P}_1 + 9\mathbf{P}_2 - (9/2)\mathbf{P}_3 + \mathbf{P}_4\big)t + \mathbf{P}_1.
\end{aligned}$$