6 Bézier Approximation

Bézier methods for curves and surfaces are popular, are commonly used in practical work, and are described here in detail. Two approaches to the design of a Bézier curve are described, one using Bernstein polynomials and the other using the mediation operator. Both rectangular and triangular Bézier surface patches are discussed, with examples.

Historical Notes

Pierre Etienne Bézier (pronounced "Bez-yea" or "bez-ee-ay") was an applied mathematician with the French car manufacturer Renault. In the early 1960s, encouraged by his employer, he began searching for ways to automate the process of designing cars. His methods have been the basis of the modern field of Computer Aided Geometric Design (CAGD), a field with practical applications in many areas.

It is interesting to note that Paul de Faget de Casteljau, an applied mathematician with Citroën, was the first, in 1959, to develop the various Bézier methods but—because of the secretiveness of his employer—never published it (except for two internal technical memos that were discovered in 1975). This is why the entire field is named after the second person, Bézier, who developed it.

Bézier and de Casteljau did their work while working for car manufacturers. It is little known that Steven Anson Coons of MIT did most of his work on surfaces (around 1967) while a consultant for Ford. Another mathematician, William J. Gordon, has generalized the Coons surfaces, in 1969, as part of his work for General Motors research labs. In addition, airplane designer James Ferguson also came up with the same ideas for the construction of curves and surfaces. It seems that car and airplane manufacturers have been very innovative in the CAGD field. Detailed historical surveys of CAGD can be found in [Farin 04] and [Schumaker 81].

6.1 The Bézier Curve

The Bézier curve is a parametric curve $\mathbf{P}(t)$ that is a polynomial function of the parameter t. The degree of the polynomial depends on the number of points used to define the curve. The method employs *control points* and produces an approximating curve (note the title of this chapter). The curve does not pass through the interior points but is attracted by them (however, see Exercise 6.7 for an exception). It is as if the points exert a pull on the curve. Each point influences the direction of the curve by pulling it toward itself, and that influence is strongest when the curve gets nearest the point. Figure 6.1 shows some examples of cubic Bézier curves. Such a curve is defined by four points and is a cubic polynomial. Notice that one has a cusp and another one has a loop. The fact that the curve does not pass through the points implies that the points are not "set in stone" and can be moved. This makes it easy to edit, modify and reshape the curve, which is one reason for its popularity. The curve can also be edited by adding new points, or deleting points. These techniques are discussed in Sections 6.8 and 6.9, but they are cumbersome because the mathematical expression of the curve depends on the number of points, not just on the points themselves.

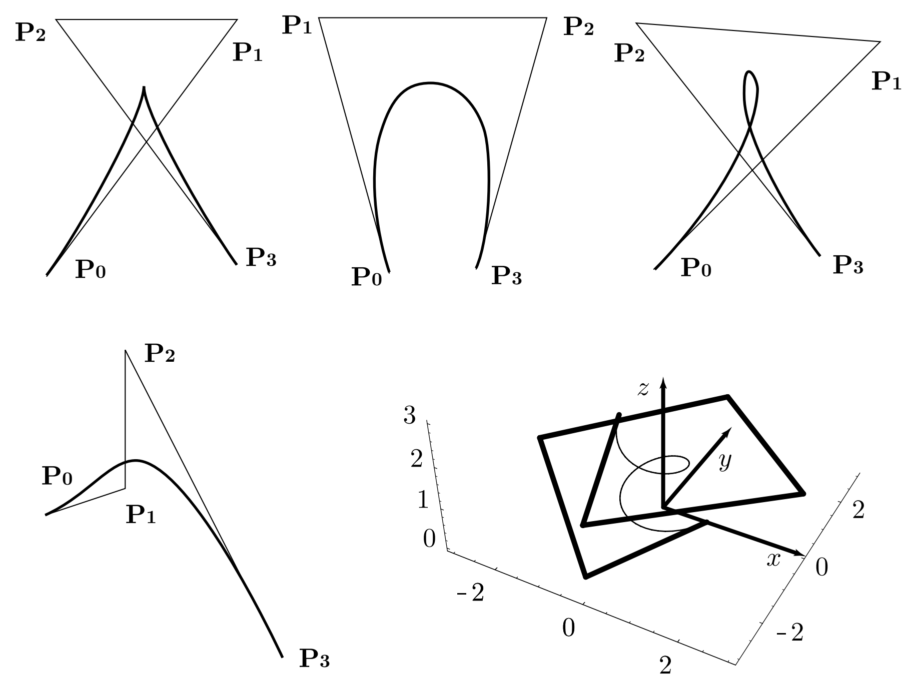

Figure 6.1: Four Plane Cubic and One Space Bézier Curves With Their Control Points and Polygons.

The *control polygon* of the Bézier curve is the polygon obtained when the control points are connected, in their natural order, with straight segments.

How does one go about deriving such a curve? We describe two approaches to the design—a weighted sum and a linear interpolation—and show that they are identical.