8
Subdivision Methods

8.1 Introduction

The Bézier curve can be constructed either as a weighted sum of control points or by the process of scaffolding. These are two very different approaches that lead to the same result. A third approach to curve and surface design, employing the process of *refinement* (also known as *subdivision* or *corner cutting*), is the topic of this chapter. Refinement is a general approach that can produce Bézier curves, B-spline curves, and other types of curves. Its main advantage is that it can easily be extended to surfaces.

8.2 Chaikin's Refinement Method

In 1974, George Chaikin came up with the idea of constructing a smooth curve from a small number of control points in several *refinement* steps. The principle of Chaikin's method is to start with a given set of control points $\mathbf{P}_i$, perform a computation that results in a new set of points $\mathbf{P}_i^1$, and repeat the process, producing more and more sets of points $\mathbf{P}_i^k$. Thus, the original control polygon is successively refined. Table 8.1 shows the notation used.

$$
\begin{array}{llll}
\mathbf{P}_0, & \mathbf{P}_1, & \ldots, & \mathbf{P}_n \\
\mathbf{P}_0^1, & \mathbf{P}_1^1, & \ldots, & \mathbf{P}_{n_1}^1 \\
\mathbf{P}_0^2, & \mathbf{P}_1^2, & \ldots, & \mathbf{P}_{n_2}^2 \\
 & & \vdots & \\
\mathbf{P}_0^k, & \mathbf{P}_1^k, & \ldots, & \mathbf{P}_{n_k}^k
\end{array}
$$

Table 8.1: Refining Control Points.

Each point $\mathbf{P}_j^k$ is computed as a weighted sum of the points $\mathbf{P}_i^{k-1}$ of the previous iteration. Thus,

$$\mathbf{P}_j^k = \sum_{i=0}^{n_{k-1}} a_{ijk}\mathbf{P}_i^{k-1} = (a_{0jk}, a_{1jk}, \ldots, a_{n_{k-1},jk}) \begin{pmatrix} \mathbf{P}_0^{k-1} \\ \mathbf{P}_1^{k-1} \\ \vdots \\ \mathbf{P}_{n_{k-1}}^{k-1} \end{pmatrix},$$

where a_{ijk} are real coefficients. Notice that each iteration produces a different number $n_k + 1$ of points. If n_k gets smaller with k, then the number of points gets smaller and smaller until a single point is left. An example is the de Casteljau scaffolding construction, a process that produces one point of the Bézier curve. At the other extreme, n_k may get larger with k, producing more points in each iteration. We then stop after a few iterations and draw the curve by drawing straight segments between the points of the last iteration. An example of this case is the Chaikin algorithm, described in example (2).

Each iteration can be completely described by its coefficient matrix

$$\begin{aligned} \begin{pmatrix} \mathbf{P}_0^k \\ \mathbf{P}_1^k \\ \vdots \\ \mathbf{P}_{n_k}^k \end{pmatrix} &= \begin{pmatrix} a_{00k} & a_{10k} & \cdots & a_{n_{k-1},0k} \\ a_{01k} & a_{11k} & \cdots & a_{n_{k-1},1k} \\ \vdots & \vdots & & \vdots \\ a_{0,n_k,k} & a_{1,n_k,k} & \cdots & a_{n_{k-1},n_k,k} \end{pmatrix} \begin{pmatrix} \mathbf{P}_0^{k-1} \\ \mathbf{P}_1^{k-1} \\ \vdots \\ \mathbf{P}_{n_{k-1}}^{k-1} \end{pmatrix} \\ &= \mathbf{M}_k \begin{pmatrix} \mathbf{P}_0^{k-1} \\ \mathbf{P}_1^{k-1} \\ \vdots \\ \mathbf{P}_{n_{k-1}}^{k-1} \end{pmatrix}, \end{aligned} \tag{8.1}$$

where $\mathbf{M}_k$ has $n_k + 1$ rows and $n_{k-1} + 1$ columns. Since the number of iterations may be large, the number of coefficients a_{ijk} may be huge. In practice, this number is significantly reduced in three ways: (1) Using a rule of calculation where most of these coefficients are zero. (2) Using coefficients a_{ij} that are independent of k. (3) Using coefficients a_{ik} that are independent of j. Case 2 is called *uniform refinement* and case 3 is termed *stationary refinement*.

Example: (1) This is the de Casteljau scaffolding construction expressed as a refinement process. The rule of refinement is

$$\mathbf{P}_j^{k+1} = 0.5(\mathbf{P}_j^k + \mathbf{P}_{j+1}^k), \tag{8.2}$$

which implies that the a_i coefficients are independent of j and k (this is a stationary uniform refinement method) and are zero except for the two coefficients a_j and a_{j+1}. The a_{ijk}'s therefore depend on i only and are given by

$$a_i = \begin{cases} 0.5, & i = j, j+1, \\ 0, & \text{otherwise.} \end{cases}$$