9
Sweep Surfaces

The surfaces described in this chapter are obtained by transforming a curve. They are not generated as interpolations or approximations of points or vectors and are consequently different from the surfaces described in previous chapters. A reader who wishes a full understanding of this chapter should be familiar with the important three-dimensional transformations (rotation, translation, scaling, reflection, and shearing) and how they are described mathematically by a 4×4 transformation matrix. This material is available in most texts on computer graphics, but the next paragraph is a short summary, for those who only need a refresher.

A three-dimensional point $\mathbf{P}=(x,y,z)$ is transformed to a point $\mathbf{P}^*=(x^*,y^*,z^*)$ by appending a fourth coordinate of 1 to it and then multiplying it by the 4×4 transformation matrix

$$\mathbf{T}=\begin{pmatrix} a & b & c & p\\ d & e & f & q\\ h & i & j & r\\ l & m & n & s \end{pmatrix}. \tag{9.1}$$

The product $(x,y,z,1)\mathbf{T}$ is a 4-tuple (X,Y,Z,H), where $H=xp+yq+zr+s$. The three coordinates (x^*,y^*,z^*) of $\mathbf{P}^*$ are obtained by dividing (X,Y,Z) by H. Hence, $(x^*,y^*,z^*)=(X/H,Y/H,Z/H)$. The top left 3×3 submatrix of $\mathbf{T}$ is responsible for scaling and reflection (parameters a, e, and j), shearing (b,c,f, and d,h,i), and rotation (all nine). The three quantities l, m, and n are responsible for translation, and s is a global scale factor. The three parameters p, q, and r are used for perspective projection.

9.1 Sweep Surfaces

A sweep surface is obtained when a space curve $\mathbf{C}(u)$, termed the *profile*, is transformed by a transformation rule $\mathbf{T}(w)$. The transformation must include translation and/or rotation and may also include scaling and shearing. We say that the surface is *swept* by the profile curve when it (the curve) is transformed. The expression of the surface is simply the product $\mathbf{P}(u, w) = \mathbf{C}(u) \cdot \mathbf{T}(w)$. The transformation $\mathbf{T}$ is a 4×4 matrix, so vector $\mathbf{C}$ should be written in homogeneous coordinates, as the 4-tuple $\mathbf{C}(u) = (x(u), y(u), z(u), 1)$.

The simplest example is the translation of a straight line. The straight segment from the origin to $(1, 0, 0)$ is given by $\mathbf{C}(u) = (u, 0, 0, 1)$ where $0 \le u \le 1$. This segment is translated along the y axis by the transformation matrix

$$\mathbf{T}(w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & w & 0 & 1 \end{pmatrix},$$

where $0 \le w \le 1$. The surface $\mathbf{P}(u, w) = \mathbf{C}(u)\cdot\mathbf{T}(w) = (u, w, 0, 1)$ swept by this segment is (after dividing by the fourth element) $\mathbf{P}(u, w) = (u, w, 0)$. This surface is simply the square, on the xy plane, whose opposite corners are the origin and point $(1, 1, 0)$.

A more interesting example is the same segment $\mathbf{C}(u) = (u, 0, 0, 1)$, where $0 \le u \le 1$, translated a distance α along the z axis while being rotated 360° about that axis. The transformation matrix is

$$\mathbf{T}(w) = \begin{pmatrix} \cos(2\pi w) & \sin(2\pi w) & 0 & 0 \\ -\sin(2\pi w) & \cos(2\pi w) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \alpha w & 1 \end{pmatrix}, \quad \text{for} \quad 0 \le w \le 1.$$

The expression of the surface is $\mathbf{P}(u, w) = (u\cos(2\pi w), u\sin(2\pi w), \alpha w)$ and it is displayed in Figure 9.1a. For $w = 0.5$, it reduces to the segment $(0, u, 0.5\alpha)$ (in the y direction), and for $w = 1$, it becomes the segment $(u, 0, \alpha)$ [a segment in the original x direction, but at a height α on the z axis].

A more general example is a rectangular surface patch constructed as a sweep surface by translating an arbitrary profile along another curve, the *trajectory*. Given the two cubic Bézier curves

$$\begin{aligned} \mathbf{C}(t) &= (1-t)^3(0, 1, 1) + 3t(1-t)^2(1, 1, 0) + 3t^2(1-t)(4, 2, 0) + t^3(6, 1, 1) \\ &= (-3t^3 + 6t^2 + 3t, -3t^3 + 3t^2 + 1, 3t^2 - 3t + 1) \\ &\text{and} \\ \mathbf{Q}(t) &= (1-t)^3(0, 0, 0) + 3t(1-t)^2(1, 2, 1) + 3t^2(1-t)(3, 2, 2) + t^3(2, 0, 1) \\ &= (-4t^3 + 3t^2 + 3t, -6t^2 + 6t, -2t^3 + 3t) \end{aligned}$$

we can create a sweep surface $\mathbf{P}(u, w)$ by translating $\mathbf{C}(u)$ along $\mathbf{Q}(w)$. The expression