4

Linear programming with set coefficients
J. Nedoma and J. Ramík

4.1 Introduction

In this chapter we investigate a linear programming problem (LP problem) already formulated in Chapter 3. Here, we consider a family of linear programming problems

\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b, \\
& \quad x \geq 0,
\end{align*}
\]

with data satisfying \( c \in c \subseteq \mathbb{R}^n, A \in A \subseteq \mathbb{R}^{m \times n}, b \in b \subseteq \mathbb{R}^m \), where \( c, A \) and \( b \) are preselected sets. In comparison to Chapter 3, here \( c, A \) and \( b \) are not necessarily (matrix or vector) intervals and the inequalities are considered in (4.1). The family of LP problems (4.1) is called the linear programming problem with set coefficients (LPSC problem). In what follows, our interest is focused on the case where \( c, A \) and \( b \) are either compact convex sets or, in particular, convex polytopes. We are interested primarily in the systems of inequalities in (4.1); later on we also deal with systems of equations. In the next section we start with the problem of duality of LPSC problems and later on we propose an algorithm, a generalized simplex method for solving such problems.

4.2 LP with set coefficients

Let \( c^T = (c_1, \ldots, c_n)^T \in c \subseteq \mathbb{R}^n, A = (a_{ij})_{i,j=1}^{m,n} \in A \subseteq \mathbb{R}^{m \times n}, b^T = (b_1, \ldots, b_m)^T \in b \subseteq \mathbb{R}^m \), where \( c, A \) and \( b \) are preselected sets. Then LPSC problem (4.1) can be rewritten as follows,
maximize \( c_1x_1 + \cdots + c_nx_n \)
subject to \( a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i, \quad i \in M, \quad x_j \geq 0, \quad j \in N. \) \hfill (4.2)

Here, \( M = \{1, 2, \ldots, m\} \) and \( N = \{1, 2, \ldots, n\}. \)

We write the LPSC problem (4.1) briefly also as

\[
\max \{c^T x | Ax \leq b, x \geq 0\}
\] \hfill (4.3)

with data satisfying \( c \in c \subseteq \mathbb{R}^n, A \in A \subseteq \mathbb{R}^{m \times n}, b \in b \subseteq \mathbb{R}^m. \)

Corresponding to Chapter 2, we consider weak and strong feasibility of LPSC problem (4.1); moreover, we introduce a new concept: strict feasibility of the LPSC problem.

Let \( m \) be a positive integer and \( U, V \) be subsets of \( \mathbb{R}^m. \) Define the following "inequality" relations,

\[
U \leq_1 V \text{ if } \forall u \in U, \forall v \in V : u \leq v, \\
U \leq_2 V \text{ if } \exists u \in U, \exists v \in V : u \leq v.
\]

For \( t \in \{1, 2\} \) we define the following sets,

\[
X_t^\leq (A, b) = \{ x \in \mathbb{R}^n \mid Ax \leq_t b, x \geq 0 \}. \tag{4.4}
\]

Here, \( Ax = \{ y \in \mathbb{R}^m \mid y = Ax, A \in A \}. \) The LPSC problem (4.2) is said to be strictly feasible if

\[
X_1^\leq (A, b) \neq \emptyset.
\]

If

\[
X_2^\leq (A, b) \neq \emptyset,
\]
we say that the LPSC problem (4.2) is weakly feasible. Moreover, we say that the LPSC problem (4.2) is strongly feasible if for each \( A \in A \) and each \( b \in b \) there exists \( x \in \mathbb{R}^n, x \geq 0, \) such that \( Ax \leq b. \)

In other words, given \( A \subseteq \mathbb{R}^{m \times n}, b \subseteq \mathbb{R}^m \) an LPSC problem (4.2) is strictly feasible if there exists a vector \( x \in \mathbb{R}^n, x \geq 0, \) such that \( Ax \leq b \) for all \( A \in A \) and all \( b \in b. \) On the other hand, an LPSC problem (4.2) is weakly feasible if there exists a vector \( x \in \mathbb{R}^n, x \geq 0, \) and there exist some data \( A \in A \) and \( b \in b \) such that \( Ax \leq b. \) An LPSC problem (4.2) is strongly feasible if for each \( A \in A \) and each \( b \in b \) there exists a vector \( x \in \mathbb{R}^n, x \geq 0, \) such that \( Ax \leq b. \)

### 4.2.1 Strict, weak and strong feasibility

In this subsection, sufficient conditions for strong feasibility of the LPSC problem are derived. We show that in the special case of the interval LP problem strict feasibility and strong feasibility are equivalent.