# 9

# THE KEPLER FACTOR

With the operator prejudice for physical theories, both objects and interactions should have an operational origin. The basic interactions, we have so far, involve in their nonrelativistic approximation the *Kepler factor* $\frac{1}{r}$. A position radial dependence $\frac{1}{r}$ arises at characteristic points: A $\frac{1}{r}$-proportional potential describes the most important nonrelativistic interactions: gravitation and electrostatics. Historically, it led to Newton's understanding of Kepler's laws by the gravitation of mass points, which was the first step to Einstein's general relativity. Furthermore, as Coulomb potential, it played a decisive role in the discovery of Maxwell's electrodynamics with its field structure, introduced by Faraday, and its extension to quantum electrodynamics and to the standard model of the electroweak gauge interactions. Its quantum-mechanical application led to the understanding of the atoms and the periodic system. In addition to nonrelativistic potentials the $\frac{1}{r}$-dependence also characterizes spherical waves $\frac{e^{iqr}}{r}$ in electrodynamics (optics).

Already for Newton the Kepler factor $\frac{1}{r}$ was geometrically motivated. The exact power $\gamma = 1$ in $\frac{1}{r^\gamma}$, not, e.g., $\frac{1}{r^{1.007}}$, is no accident. The square of the Kepler factor is, up to a constant, the inverse of the 2-sphere area $|\Omega^2(r)| = 4\pi r^2$. $\frac{e^{\kappa r}}{r}$ will be interpreted as the 2-sphere spread for a coefficient $e^{\kappa r}$ of a representation of position translations where $\kappa$ can be imaginary, $\kappa = \pm i|\vec{q}|$, for scattering waves or negative $\kappa = -|Q|$ for bound waves or trivial in $\frac{1}{r}$.

The small-distance singularity of the Kepler factor for $r = 0$ is known to prevent an interpretation of the electron mass as of electromagnetic origin. It is the nonrelativistic precursor of the divergences arising, e.g., in Feynman integrals in quantum theory for particle fields (chapter "Spectrum of Spacetime").

A Kepler potential is the infinite range $\ell \to \infty$ limit of a *Yukawa potential* with $\frac{e^{-\frac{r}{\ell}}}{r}$, which, with respect to second order differential equations with $\vec{\partial} = \frac{\partial}{\partial \vec{x}}$, is the special relativistic position supplement of an irreducible causal time representation $d_t = \frac{d}{dt}$ with frequency $\mu$:

$$\frac{e^{-\frac{r}{\ell}}}{r} \leftrightarrow \sin|t\mu| \text{ since } \begin{cases} (d_t^2 + \mu^2) & \frac{\sin|t|\mu}{\mu} & = 2\delta(t), \\ (-\vec{\partial}^2 + \frac{1}{\ell^2}) & \frac{e^{-\frac{r}{\ell}}}{2\pi r} & = 2\delta(\vec{x}). \end{cases}$$

Also, such a space-time parallelism determines the power in the Kepler potential $\frac{1}{r^\gamma}$ to be exactly $\gamma = 1$. If the frequency and the potential range combine

the speed of light $\mu\ell = c$, the equations above arise as time and position projections of a Lorentz-invariant inhomogeneous Klein-Gordon equation with the infinite range potential related to a mass-zero structure.

The irreducible unitary representations of time $\mathbb{R} \ni t \longmapsto e^{iEt} \in \mathbf{U}(1)$ in quantum mechanics that decompose the action of the Hamiltonian with energy $E$ eigenvalues induce a Hilbert space structure. The inclusion of position representations spreads the state vectors to position orbits (Schrödinger wave functions) $\mathbb{R}^3 \ni \vec{x} \longmapsto \psi_E(\vec{x}) \in \mathbb{C}$. The radial translations $r = |\vec{x}|$ as rotation invariant part constitute the positive cone $\mathbb{R}_+$ of a 1-dimensional noncompact group $\mathbb{R}$ which, for a potential $V(r)$, comes in representations with the eigenvalues determined by the difference $E - V$. There arise scattering waves for kinetic energy $E - V > 0$ (compact position representations) and imaginary radial translation eigenvalue $\pm i|\vec{q}| = \pm i\sqrt{2(E-V)}$ (real momentum) and bound waves for $E - V < 0$ (noncompact position representations) with strictly negative eigenvalue $-|Q| = -\sqrt{-2(E-V)}$ (imaginary "momentum") and binding energy $|E - V|$.

## 9.1 Center of Mass Transformation

A Lagrangian determines the time development of two classical mass points in Euclidean position space with dual position-momentum pairs $(\vec{x}_i, \vec{p}_i)_{i=1,2}$ by a Hamiltonian $H$

$$L(1,2) = \vec{p}_1 d_t\vec{x}_1 + \vec{p}_2 d_t\vec{x}_2 - H(1,2), \quad H(1,2) = \tfrac{\vec{p}_1^2}{2m_1} + \tfrac{\vec{p}_2^2}{2m_2} + V(\vec{x}_1, \vec{x}_2).$$

The time action is induced by the gravitational interaction with Newton's potential and by the electrostatic interaction with Coulomb's potential:

$$V(\vec{x}_1, \vec{x}_2) = \tfrac{g_0}{|\vec{x}_1 - \vec{x}_2|}, \quad g_0 = \left\{ \begin{array}{ll} -G_N m_1 m_2, & \text{Newton's constant } G_N \\ & \text{with masses } m_{1,2}, \\ \tfrac{1}{4\pi\epsilon_0} Q_1 Q_2, & \text{vacuum dielectricity constant } \epsilon_0 \\ & \text{and charges } Q_{1,2}. \end{array} \right.$$

If the potential depends of the mass point distance only $V(\vec{x}_1, \vec{x}_2) = V(\vec{x}_1 - \vec{x}_2)$, the center of mass dynamics can be separated by an orthogonal transformation of the two momenta, leaving invariant the kinetic energy, and the contragredient transformation of the positions as dual variables:

$$\left.\begin{array}{rl} \tfrac{\vec{p}_1^2}{m_1} + \tfrac{\vec{p}_2^2}{m_2} & = \tfrac{\vec{P}^2}{m} + \tfrac{\vec{p}^2}{M} \\ \vec{p}_1 d_t\vec{x}_1 + \vec{p}_2 d_t\vec{x}_2 & = \vec{P} d_t\vec{X} + \vec{p} d_t\vec{x} \end{array}\right\} \iff \left\{ \begin{array}{rl} O(\beta)\begin{pmatrix} \frac{\vec{p}_1}{\sqrt{m_1}} \\ \frac{\vec{p}_2}{\sqrt{m_2}} \end{pmatrix} & = \begin{pmatrix} \frac{\vec{p}}{\sqrt{m}} \\ \frac{\vec{P}}{\sqrt{M}} \end{pmatrix}, \\ \check{O}(\beta)\begin{pmatrix} \sqrt{m_1}\vec{x}_1 \\ \sqrt{m_2}\vec{x}_2 \end{pmatrix} & = \begin{pmatrix} \sqrt{m}\vec{x} \\ \sqrt{M}\vec{X} \end{pmatrix}, \end{array}\right.$$

$$\text{with } O(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} = O^{-1T}(\beta) = \check{O}(\beta) \in \mathbf{SO}(2);$$

$\sqrt{\mu}$ and $\frac{1}{\sqrt{\mu}}$ for the masses $\mu \in \{m_1, m_2, m, M\}$ arise as normalizations, inverse to each other, for the corresponding dual position-momentum pairs.

The distance dependence determines the rotation angle

$$\vec{x} \sim \vec{x}_1 - \vec{x}_2 \Rightarrow \tan^2\beta = \tfrac{m_1}{m_2}.$$