1

SPACETIME TRANSLATIONS

An immediate “naive” description and quantification of the physical phenomena in our temporal and spatial neighborhood leads to real linear spacetime structures as a first mathematical formalization: the spacetime translation vector space with its affine operations. From the Galilean-Newtonian mechanics up to Einstein’s special relativity, real translations with the possibility to distinguish absolute neutral points - “now” for time and “here” for position - have played the most important role. Up to today, an interpretation of a dynamics by its experimental predictions uses decisively the concepts “mass” and “angular momentum” (spin, polarization). They characterize invariant properties with respect to spacetime translations and rotations respectively as operations in the Poincaré group.

In the interpretation of Leibniz, time and position are relational concepts, to describe and to quantify the behavior and properties of objects with respect to operations defining a dynamics and related experiments. The causal structure of spacetime operations are their most important feature. Newton’s successful interpretation of time and position space as having an absolute ontology - “God-given temporal and spatial boxes for the dynamical objects” - is not favored here.

Time and position are operations. The naive interpretation of the space coordinates in classical physics as describing directly the position of a point particle gave way to the interpretation of space as a reservoir for parametrizing operations by three real numbers. Such numbers are taken from the spacetime manifold of relativity. The position coordinates in quantum-mechanical wave functions or in particle quantum fields parametrize position operations, especially translations for interaction-free structures. With the operator structure of quantum theory, the concept of a point particle has, if at all, only a significance as one experimental projection.

In this chapter, the concepts of order (causality), linear duality, and isomorphic dual linear spaces are considered for three pairs of physically relevant vector spaces: for “time translations with energies”, for “position translations with momenta,” and for “spacetime translations with energy-momenta.” Energies and momenta are the respective eigenvalues for time and position translations. The vector spaces come with action groups, with reflections, rotations, and Lorentz transformations, as invariance groups for the metrics

that define the self-duality for dual pairs. On this level time and space are, operationally, abelian Lie algebras. An operational connection of the dual position-momentum structures comes with the nonabelian Heisenberg Lie algebra $[\mathbf{x}_a, \mathbf{p}^b] = \delta_a^b \mathbf{I}$ (chapter "Quantum Algebras").

1.1 Time Translations

A linear time model $\mathbb{T}$ ("tempus") collects the *time translations* into a real 1-dimensional vector space. Time vectors should be considered as operations, not as "inert absolutely given points." "Time" in this chapter is understood always as a vector space containing the time translations. The trivial translation $0 \in \mathbb{T}$ ("now") is distinguished as the neutral element of the additive group. The spaces on which time translations act will be discussed in the chapter "Time Representations."

For time, formalized by the real numbers $\mathbb{R}$ as a 1-dimensional vector space, general vector space concepts like basis, dual space, scalar product, invariances, and topology, appear academically blown up: their importance becomes clearer if applied afterward to real 3-dimensional position translations or if time and position are embedded into a 4-dimensional vector space with spacetime translations.

Each *time translation basis*[1] gives rise to an isomorphism $x \leftrightarrow x_0$ between time and the real numbers as time coordinates

$$\mathbb{T} = \{x = x_0 \mathbf{p}^0 \bigm| x_0 \in \mathbb{R}\} \cong \mathbb{R}, \quad \text{basis: } \{\mathbf{p}^0\}.$$

With that, all $\mathbb{R}$-structures, called *natural*, are transportable to time or vice versa: The concept of the ordered natural numbers with their rational and real extensions may be considered to be an abstraction of the time structures. The characterization of time translations by real numbers uses the linear forms of time that constitute the dual time $\mathbb{T}^T = \{\mathbb{T} \longrightarrow \mathbb{R}\}$, the *frequency or energy space* with the eigenvalues of the time translations. A time basis $\{\mathbf{p}^0\}$ comes with a unique dual basis $\{\mathbf{x}_0\}$ of the frequency space

$$\begin{aligned} &\mathbb{T}^T = \{p = p^0 \mathbf{x}_0 \bigm| p^0 \in \mathbb{R}\} \cong \mathbb{R}, \quad \text{dual bases: } \langle \mathbf{x}_0, \mathbf{p}^0 \rangle = 1, \\ &\mathbf{x}_0 : \mathbb{T} \longrightarrow \mathbb{R}, \quad \mathbf{x}_0(x) = x_0. \end{aligned}$$

The manifold of all time bases can be obtained by operating with the general linear group on one fixed basis, i.e., by multiplying by a nontrivial scalar

$$\mathbf{GL}(\mathbb{R}) \bullet \{\mathbf{p}^0\} \cong \mathbf{GL}(\mathbb{R}) \cong \mathbb{R}^\diamond = \{\alpha \in \mathbb{R} \bigm| \alpha \neq 0\}.$$

With a basis, time is *totally ordered* via the real coefficients

$$x = x_0 \mathbf{p}^0 \succeq_{\mathbf{p}} 0 \iff x_0 \geq 0.$$

[1] In this chapter, basis vectors for time and position are denoted with boldface letters.