2

TIME REPRESENTATIONS

A dynamics can be characterized and will be understood as an action of time and position which, together with time and position translations as their tangent structures, are modeled by operations from real Lie groups and Lie algebras respectively. Time and position operations come in realizations and representations acting on sets and vector spaces. The solution of a dynamics is the decomposition of the time and position representations involved into nondecomposable, perhaps even irreducible, representations. Representations are characterized by invariants (masses, spins, etc.) and eigenvalues (energies, momenta, helicity, etc.) for the operations that define the properties of physical objects. With this program, physical theories become to a great extent applied representation and realization theory.

In classical dynamics the time realizations are visible in the mass point orbits $t \longmapsto \mathbf{x}(t)$ in position as solutions of the equations of motion with a fixed energy, imposed by initial or boundary conditions. In quantum mechanics time and position orbits are information-valued ("probability amplitudes"). They are given, e.g., by Schrödinger wave functions $(t, \vec{x}) \longmapsto \Psi(t, \vec{x}) = e^{iEt}\psi_E(\vec{x})$, which are time and position orbits in a complex Hilbert space with probability interpretation (chapter "The Kepler Factor").

In this chapter, the complex finite-dimensional representations of the "simplest" nontrivial real 1-dimensional Lie group $\mathbf{D}(1) = \exp \mathbb{R}$ and its Lie algebra $\log \mathbf{D}(1) = \mathbb{R}$ are considered. Its structures can be formulated in a time-related language: The simplest, and also characteristic, examples are given by the free Newtonian mass point with mass M and the harmonic oscillator with string constant k with Hamiltonians $H_{1,0}$ and the equations of motion for position-momentum $(\mathbf{x}, \mathbf{p})$:

$$\begin{array}{lll} H_1 = \frac{\mathbf{p}^2}{2M} & \Rightarrow \frac{d\mathbf{x}}{dt} = \frac{\mathbf{p}}{M}, & \frac{d\mathbf{p}}{dt} = 0, \\ H_0 = \frac{\mathbf{p}^2}{2M} + \frac{k}{2}\mathbf{x}^2 & \Rightarrow \frac{d\mathbf{x}}{dt} = \frac{\mathbf{p}}{M}, & \frac{d\mathbf{p}}{dt} = -k\mathbf{x}. \end{array}$$

$\mathbf{D}(1)$ will be called the time group or, in special relativity, eigentime group. In larger operation groups, e.g., in spacetime, the general name "causal group" is appropriate.

To represent real Lie operations on complex vector spaces, such a space has to come with a conjugation, definite or indefinite: Real groups have to be

represented in the complex by unitary automorphisms.[1] The conjugation for a time representation implements the time reflection and determines an inner product of the complex representation space, which, for a positive conjugation, is the origin for the scalar product, leading in quantum theories to "probability amplitudes" for the interpretation of experiments (chapter "Quantum Probabilities").

Obviously from a general group theoretical point of view, the representations of the two real 1-dimensional abelian Lie groups $\mathbf{D}(1) = \exp \mathbb{R}$ and $\mathbf{U}(1) = \exp i\mathbb{R}$ as noncompact and compact subgroups of the complex group $\mathbf{GL}(\mathbb{C}) = \mathbb{C}^\diamond = \exp \mathbb{C}$ are the basic ingredients for the representations of all real and complex Lie groups that contain "many" $\mathbf{D}(1)$ and $\mathbf{U}(1)$-isomorphic subgroups.

2.1 The Time Group

The totally ordered noncompact real 1-dimensional Lie group $\mathbf{D}(1)$ is used as a time model. This abelian group can be written multiplicatively as $\mathbf{D}(1)$ or additively as $\mathbb{R}$:

$$\mathbf{D}(1) = \exp \mathbb{R} = \{e^t \bigm| t \in \mathbb{R}\} \cong \mathbb{R} = \log \mathbf{D}(1).$$

The Lie group isomorphism is given by the exponential and the logarithm.

The classes with respect to the discrete subgroup of the integers $\exp \mathbb{Z} \cong \mathbb{Z}$ constitute the quotient group $\exp \mathbb{R}/\exp \mathbb{Z}$, isomorphic to the *compact* additive group $\mathbb{R}/\mathbb{Z}$ of the reals modulo the integers, i.e., to the 1-dimensional torus. As a real Lie group it is isomorphic to the unit circle $\mathbf{U}(1)$, the phase group of the complex numbers:

$$\mathbf{U}(1) = \exp i\mathbb{R} = \{e^{i\alpha} \bigm| 0 \le \alpha < 2\pi\} \cong \mathbb{R}/\mathbb{Z}, \quad \log \mathbf{U}(1) = i\mathbb{R}.$$

All connected Lie groups with $\mathbb{R}$-isomorphic Lie algebra arise from the simply connected Lie group $\mathbf{D}(1)$ by the classes with respect to the discrete normal subgroups. Since $\exp \mathbb{Z} \cong \mathbb{Z}$ is, up to isomorphism, the only nontrivial closed subgroup, only $\mathbf{D}(1)$ and $\mathbf{U}(1)$ occur as images of nontrivial time $\mathbf{D}(1)$-representations.

Thus three types of time orbits are possible: the trivial representation with an 1-elementic orbit $\mathbb{R}/\mathbb{R} \cong \{1\}$ and two 1-dimensional orbits, as manifolds isomorphic either to the circle $\mathbb{R}/\mathbb{Z} \cong \mathbf{U}(1)$ or to the real line $\mathbb{R}$. This is illustrated (chapter "The Kepler Factor") in a classical description by the solar system, with hyperbolic orbits for never-returning comets, elliptic ones for planets, and the trivial orbit for the sun (more exactly, for the center of mass) with trivially represented $\mathbb{R}$-subgroups $\{0\}$, $\mathbb{Z}$, and $\mathbb{R}$. Quantum-mechanical energy eigenstates are $\mathbf{U}(1)$-orbits in the Hilbert space under question.

[1] Orthogonality and unitarity can be definite or indefinite, e.g., $\mathbf{O}(1,3)$ or $\mathbf{O}(4)$ and $\mathbf{U}(1,3)$ or $\mathbf{U}(4)$.