3

SPIN, ROTATIONS, AND POSITION

In addition to time with its order structure, formalized by the real Lie group $\mathbf{D}(1)$ (causal group) and its Lie algebra $\mathbb{R}$ (time translations) with energy (frequency) characterizing its eigenvectors, a physical dynamics also represents position with the related operations: The real 3-dimensional position translations come with a scalar product, invariant under rotations. The spatial form of an object, a sphere, an ellipsoid, a cube, a tree, etc., is characterized by its possibly very complicated properties with respect to rotations (chapter "Harmonic Analysis"). There is only one Lie algebra[1] of real dimension three, which Cartan called A_1^c, that realizes the position operations. The exponent of this compact Lie algebra is $\mathbf{SU}(2) = \exp A_1^c$, the *spin group*, whose classes with respect to its discrete center $\mathbb{I}(2) \cong \{\pm \mathbf{1}_2\}$ are the rotations $\mathbf{SO}(3) \cong \mathbf{SU}(2)/\mathbb{I}(2)$. The invariant property for these operations is angular momentum or spin. The abstract $\mathbf{SU}(2)$-operations are realized in basic physical interactions and particles, e.g., by spin, by isospin, and in Lorentz transformations. In the following the physically suggestive position- and spin-oriented language will be used.

In this chapter all finite-dimensional representations of the complex 3-dimensional Lie algebra A_1 and all representations of its compact form A_1^c are considered. From the group-theoretical point of view, the "simplest simple" Lie algebra A_1 is fundamental for all nonabelian, especially semisimple, Lie operations that, in some sense can be considered as "lumped together" spin structures (chapter "Simple Lie Operations"). In the representations of the nonabelian $\mathbf{SU}(2)$, the integer winding numbers $Z \in \mathbb{Z}$ for the representations $e^{i\alpha} \longmapsto e^{iZ\alpha}$ of the abelian circle group $\mathbf{U}(1) = \exp i\mathbb{R}$ (1-torus) come in reflected winding number pairs $(Z, -Z)$ for the dual $\mathbf{U}(1)$-subrepresentations in $\mathbf{SO}(2)$. The representation characteristic maximal natural number $|Z_{\max}| = 2J$ defines the spin J with the $\mathbf{SU}(2)$-representation space dimension $1 + 2J$, e.g., the position dimension $s = 3$ for the adjoint representation $J = 1$.

[1] In this chapter a Lie algebra structure of a vector space L is defined up to linear equivalence, i.e., up to vector space automorphisms $\mathbf{GL}(L)$.

3.1 Linear Operations on the Alternative

The spin structure can be motivated by discrete noncommutative operations: Such operations are defined, minimally, as acting on a 2-element set $\{\circ, \bullet\}$, a *basic alternative*, by an exchange $\bullet \leftrightarrow \circ$ ("shut $\leftrightarrow$ open" or "up $\leftrightarrow$ down") in addition to the identity $(\bullet, \circ) \leftrightarrow (\bullet, \circ)$ ("nothing changes").

To algebraize this structure: A *numerical valuations* of the operations on the two-element set $\{\circ, \bullet\}$ is the free vector space $K^{\{\circ,\bullet\}}$ with the mappings of the basic alternative $\{\circ, \bullet\} \longrightarrow K$ into a number field, i.e., a 2-dimensional linear space $V \cong K^2$. The simplest field $\mathbb{Z}_2 = \{0, 1\}$ allows a valuation with the truth values "false" and "true" ("bit"), the embedding into real and then complex numbers $\mathbb{R}$ and $\mathbb{C}$ allows extended valuations (modalities) with probabilities and even "probability amplitudes" (chapter "Quantum Probability")

$$\begin{array}{lccccc} \text{field } K: & \mathbb{Z}_2 & \subset & \mathbb{R} & \subset & \mathbb{C}, \\ \text{valuation:} & \{0,1\} & \subset & [0,1] & \subset & \mathbf{U}(1) \times [0,1]. \end{array}$$

With the alternative formalized by two basic vectors of a free vector space K^2,

$$\left.\begin{array}{rlllll} V\text{-basis:} & \circ = & e^1 \simeq \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \bullet = & e^2 \simeq \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ V^T\text{-basis:} & & \check{e}_1 \simeq (1,0), & & \check{e}_2 \simeq (0,1) \end{array}\right. , \quad \text{dual } \langle \check{e}_A, e^B \rangle = \delta_A^B.$$

it can be acted on by the transitions $\{\sigma_+, \sigma_-\}$

$$\begin{array}{llll} \sigma_+ & = e^1 \otimes \check{e}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \sigma_+(e^1) = 0, & \sigma_+(e^2) = e^1, \\ \sigma_- & = e^2 \otimes \check{e}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & \sigma_-(e^1) = e^2, & \sigma_-(e^2) = 0. \end{array}$$

The sum $\sigma^1 = \sigma_+ + \sigma_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents the exchange operation $\bullet \leftrightarrow \circ$.

From now on the field K is chosen to be complex or real. The operations $\sigma_\pm$ together with their diagonal commutator σ_0 constitute a basis for the 3-dimensional operation Lie algebra of the alternative

$$\begin{array}{l} [l_-, l_+] = h, \quad [h, l_\pm] = \pm 2 l_\pm, \\ l_\pm \simeq \pm\sigma_\pm, \quad h \simeq \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{array}$$

The complex linear combinations of the three elements $\{l_\pm, h\}$ as *spherical Weyl (Cartan) basis* with the Lie brackets above define the Lie algebra $A_1 \cong \mathbb{C}^3$. There exist *Cartesian (Euclidean) bases* $\{l^a\}_{a=1}^3$ with totally antisymmetric structure constants $\epsilon_c^{ab} = -\epsilon^{abc}$, $\epsilon^{123} = 1$,

$$\begin{array}{l} l^1 = \frac{l_- - l_+}{2i}, \quad l^2 = \frac{l_- + l_+}{2}, \quad l^3 = i\frac{h}{2}, \\ il_+ = l^1 + il^2, \quad -il_- = l^1 - il^2, \quad ih = 2l^3, \\ [l^a, l^b] = \epsilon_c^{ab} l^c = -\epsilon^{abc} l^c. \end{array}$$

The real span of a Cartesian basis defines the lowest-dimensional compact Lie algebra $A_1^c \cong \mathbb{R}^3$ (supindex c for compact) as compact form of A_1, isomorphic