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ANTISTRUCTURES: The Real in the Complex

Quantum theory is a real theory, formulated with unitary operations, e.g. with $\mathbf{U}(1)$ or $\mathbf{SU}(n)$, which are real Lie groups acting on complex spaces with a conjugation. Complex structures with a conjugation have to be seen as doubled real structures, i.e., $\mathbb{C}_{\mathbb{R}} = \mathbb{R} + i\mathbb{R}$.

Complex numbers have two physically important properties: First, the involutive canonical conjugation implements nontrivially the *future-past reflection* $\overline{\alpha} \stackrel{\mathtt{T}}{\leftrightarrow} \alpha$, $\mathtt{T} = \star$, i.e., the reflection of the causal order, for complex vector spaces in the suggestive bra-ket notation $\langle v| \stackrel{\mathtt{T}}{\leftrightarrow} |v\rangle$. The real in the complex is established with a conjugation-induced sesquilinear form, e.g., with a scalar product (probability amplitudes) which in quantum theory leads to probabilities to describe experiments. The probabilities as products, for the numbers $\langle \alpha|\alpha\rangle = \overline{\alpha}\alpha$ or in the past-future connecting scalar products $\langle v|v\rangle$, are positive definite. Second, by the algebraic completeness of the complex numbers, there exist eigenvalues and eigenvectors for all complex linear transformations and eigenvector bases for semisimple finite-dimensional complex linear transformations. The irreducibility of real degree-2 polynomials, e.g., of X^2+1, is reflected in the real nondiagonalizability, e.g., of the harmonic oscillator time translation matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: In the real, the Hamiltonian $H = \frac{p^2+x^2}{2}$ of a harmonic oscillator has no time translation eigenvalues and no eigenvectors. For the same reason, the rotation group $\mathbf{SO}(3)$ acting on a real 3-dimensional vector space, e.g., on the position translations, also has no nontrivial diagonalizable subgroup, e.g., there do not exist eigenvectors with nontrivial eigenvalues for a maximal abelian subgroup $\mathbf{SO}(2)$, e.g., for the generating angular momentum $\mathcal{O}^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. If objects are defined as eigenvectors with respect to time and spin group action, both modeled by real Lie groups, a complex formulation is necessary.

Even the canonical conjugation of the numbers has its intricacies: It cannot be used for the definition of the reals as its invariants[1] since it is not uniquely

[1] Any involutive field automorphism keeps the rationals $\mathbb{Q}$ fixed. Since the natural $\mathbb{C}$-topology is determined from the natural $\mathbb{R}$-topology, topological arguments cannot be used to define $\mathbb{R}$ in $\mathbb{C}$.

determined by the property to be a nontrivial involutive automorphism of the complex field $\mathbb{C}$; there are infinitely many, e.g., one with

$$\sqrt[4]{2} \leftrightarrow i\sqrt[4]{2} \Rightarrow \sqrt{2} \leftrightarrow -\sqrt{2} \Rightarrow 2 \leftrightarrow 2,$$

whose existence can be proved by general arguments (involving the axiom of choice) but whose explicit form is unknown. The complex numbers with the canonical conjugation are a real 2-dimensional algebra $\mathbb{C}_\mathbb{R} = \mathbb{R} \oplus i\mathbb{R}$, e.g., as real Clifford algebra (chapter "Quantum Algebras").

Vector spaces over a field K have K-linear mappings as morphisms. Conjugate linear mappings of complex vector spaces, i.e., $f(\alpha v) = \overline{\alpha} f(v)$, are real linear. Therefore conjugate linear (antilinear) mappings play a role for complex representations of real structures, and only there.

Even with the canonical number conjugation, the conjugation for vector spaces is not unique; concepts like Hermitian and unitary require the specification of the conjugation they are defined with, in the mathematical literature called a *complex structure*. With the real isomorphism for the complex numbers $\mathbb{C}_\mathbb{R} \cong \mathbb{R}^2$, complex n-dimensional representations are real $2n$-dimensional. There is no natural isomorphism of a complex n-dimensional space with $n \geq 2$ to one of its $2n$-dimensional real forms, and therefore in general no natural realization of real structures in complex spaces. For a vector space $V \cong \mathbb{C}^n$, there are different conjugation types. They are characterized by different signatures in $\mathbf{U}(p,q)$, $p+q=n$, starting nontrivially for two dimensions with definite $\mathbf{U}(2)$ and indefinite $\mathbf{U}(1,1)$. All possible conjugations and all real forms are taken account of by using two complex vector spaces (space and antispace), canonically conjugated to each other. The induced canonical conjugation ("anticonjugation") doubles the complex scheme with duality to a quartet structure.

The mathematical discussion of antistructures, although rather formal, has important physical applications: The anticonjugation induced doubling arises most prominently in the "particle-antiparticle" dichotomy, which is extensively used in quantum fields (chapter "Massive Particle Quantum Fields"). The nontrivial action of the Lorentz group on finite-dimensional complex representation spaces, necessarily with dimension $n \geq 2$, is formulated with doubling anticonjugation. This is visible in the transition from one Pauli representation for spin $\mathbf{SU}(2)$ to the two left- and right-handed Weyl representations for Lorentz $\mathbf{SL}(\mathbb{C}^2)$ (chapter "Lorentz Operations"). More generally, the anticonjugation gives a natural doubling of representations of real Lie groups (chapter "Simple Lie Operations").

4.1 Anticonjugation

A vector space is an additive group with a compatible scalar multiplication. For each vector space V with complex scalar multiplication, there is its *antispace (canonical conjugated vector space)* $\overline{V} \in \underline{\mathbf{vec}}_\mathbb{C}$, defined by the same additive group, but equipped with a scalar multiplication, canonically conjugated in