5

SIMPLE LIE OPERATIONS

Operators in quantum theory come as linear transformations. They are characterized by invariants and eigenvectors with eigenvalues. The numerical results for a measurement of physical properties produce eigenvalues of operations. Given a set of operators, one will ask for a maximal subset that allows a simultaneous measurement for the eigenvalues of the operators therein. This involves the problem to find common eigenvectors or, for matrices, a common diagonal, i.e., a maximal common abelian structure, which for quantum operations may be called a classical projection. Such considerations for a Lie algebra of operators lead to the concept of a *Cartan subalgebra*, which is mathematically important for the classification and the representation of Lie algebras. An exhaustive physical measurement is mathematically formalized as a maximal diagonalization ("Cartanization").

Representations of Lie algebras and Lie groups are characterized by eigenvalues with the associated principal spaces (generalized eigenspaces with eigenvectors and, possibly, also nilvectors) of the endomorphisms involved. As seen in the triagonal Jordan form for principal vector bases, one endomorphism of a complex vector space is characterized by its eigenvalues, their degeneracy, order and multiplicity. A set of complex endomorphisms that constitutes a solvable Lie algebra can be brought simultaneously to a triagonal matrix form: there exist common eigenvectors with the eigenvalues constituing weights i.e., Lie algebra forms. For an even nilpotent complex Lie algebra of operations, the direct decomposition of the space acted on into weight-related principal spaces is possible, i.e., the endomorphisms have a basis of common principal vectors.

A Cartan subalgebra of a Lie algebra is a maximal nilpotent Lie subalgebra. A Lie algebra representation is characterizable by eigenforms (weights) of a Cartan subalgebra. The representation vector space allows a spectral decomposition into principal spaces with respect to a Cartan Lie subalgebra. For semisimple Lie algebras the Cartan subalgebras are even abelian; their representations are diagonalizable. In this case, the representation space can be spanned by simultaneous eigenvectors of a Cartan subalgebra; there are no nilvectors; the representation vector space allows a spectral projector decomposition without nilpotents. This leads - for finite dimensions - to the *Cartan classification of semisimple Lie algebras* (this chapter) und their

representations (next chapter), which will be seen to be a beautiful theory after one has become familiar with the initially rather complicated-looking concepts involved.

Throughout this chapter all vector spaces are assumed to be finite $\mathbb{K}$-dimensional. With Ado's theorem finite-dimensional Lie algebras can be represented by finite matrices.

5.1 Diagonalization of Operations

5.1.1 Eigenspaces and Eigenforms (Weights)

An important concept for operations are weights as a generalization and collection of eigenvalues. To remember: Each endomorphism f of a vector space $V \cong \mathbb{K}^D$, $D \geq 1$, is a unique sum of a semisimple and a nilpotent endomorphism $f = h + n \sim \left(\begin{array}{c|c} h_1 & n \\ \hline 0 & h_2 \end{array}\right)$. For the algebraically closed complex numbers, it is *triagonalizable*, i.e., there exist appropriate bases for a Jordan matrix with the eigenvalues $\{\alpha_a\}_{a=1}^m$ on the diagonal. A complex endomorphism h is even *diagonalizable* iffsemisimple, i.e., iff h has only degree-1 factors $(X - \alpha)$ in the minimal polynomial.

Generalizing from one endomorphism f and therefore from a 1-dimensional operation space $\mathbb{K}f$, one analyzes the *action of an operator vector space* $W \cong \mathbb{K}^r$, $r \geq 1$, consisting of endomorphisms $W \subseteq \mathbf{AL}(V)$ of $V \cong \mathbb{K}^D$. A common eigenvector v for two endomorphisms $f^{1,2}(v) = w^{1,2}v$ with eigenvalues $w^{1,2}$ (for more than one endomorphism α is replaced by w) is an eigenvector for all linear combinations $\beta_1 f^1 + \beta_2 f^2$ with the corresponding linear combinations $\beta_1 w^1 + \beta_2 w^2$ as eigenvalues, i.e., the eigenvalues arise from the action of a linear form $w \in W^T$ (eigenform) on the operator vector space:

$$\langle w, \beta_1 f^1 + \beta_2 f^2 \rangle = \beta_1 \langle w, f^1 \rangle + \beta_2 \langle w, f^2 \rangle = \beta_1 w^1 + \beta_2 w^2.$$

Now the general definition: If for a linear form $w : W \longrightarrow \mathbb{K}$ the *eigenvector space, common for all endomorphisms from* W,

$$\begin{aligned} &V_w(W) = \{v \in V \mid f(v) = \langle w, f \rangle v \text{ for all } f \in W\} \in \underline{\mathbf{vec}}_{\mathbb{K}}, \\ &w \in W^T, \quad V_w(W) = \bigcap_{f \in W} V_{\langle w, f \rangle}(f), \end{aligned}$$

does not consist only of the trivial vector $0 \in V$, the vector space $V_w(W)$ is called the *eigenspace for the weight (eigenform)* $w \in W^T$ *of the operators* W *on* V. The eigenspace $V_w(W)$ consists of the vectors $v \in V$ that are eigenvectors for *all* operators $f \in W$ with an operator dependent eigenvalue $\langle w, f \rangle \in \mathbb{K}$. Then v is a W-eigenvector iff v is an eigenvector for a W-basis $\{f^j\}^{j=1,\dots,r}$. The eigenvalue $\alpha = \langle w, f \rangle$ for *one* operator f and thus for a 1-dimensional operator space $\mathbb{K}f$ is generalized to the weight (eigenform) w for the operator space W. The *multiplicity* of a weight w is the dimension of its eigenspace:

$$M_w = \dim_{\mathbb{K}} V_w(W) \leq \dim_{\mathbb{K}} V.$$