6

RATIONAL QUANTUM NUMBERS

Physical objects, basically bound and scattering waves or elementary particles, are "collections of properties" of quantum operations, i.e., vectors with invariants and eigenvalues. That all the real numbers resulting from physical measurements can be interpreted by discrete or continuous spectra of real Lie operations is, perhaps, the strongest argument for the relevance of the Lie group approach to physics. Obviously, it is principally undecidable, because of the error bars, if the experimental numbers yielding spectra are rational or continuous. On the one hand, the electric charge number and the spin numbers seem to come from a discrete rational spectrum. On the other hand, energy and momenta for scattering states seem to come from a continuous real spectrum and probably also the masses of elementary particles.

All discrete *rational* invariants and eigenvalues, quantum numbers in the original sense ("natura facit saltus"), can be related to representations of *compact* groups, sometimes as subgroups of noncompact groups, e.g., $\mathbf{SO}(2) \subset \mathbf{SO}(1,2)$. Continuous quantum numbers come only from noncompact operations with their faithful infinite-dimensional Hilbert space representations (chapter "Harmonic Analysis").

There occur mixed situtations: Energy levels for nonrelativistic bound states start from irreducible, not faithful representations of time $\mathbb{R} \ni t \longmapsto e^{iEt} \in \mathbf{U}(1)$ with a continuous invariant $E \in \mathbb{R}$. The discrete energy levels, e.g., in harmonic oscillators (chapter "Quantum Algebras") or atomic bound states (chapter "The Kepler Factor"), are related to tensor representations as integer powers of the defining one, e.g., $\mathbf{U}(1) \ni e^{iEt} \longmapsto (e^{iEt})^z \in \mathbf{U}(1)$, $z \in \mathbb{Z}$.

Ultimately, the representation theory of compact groups is simple (not trivial); it is connected, via the Cartan subgroups, to the representations of the compact 1-dimensional Lie group $\mathbf{U}(1) \cong \exp i\mathbb{R}$ (circle, 1-dimensional torus) whose irreducible ones are characterized by integer winding numbers (e.g., charge numbers) as powers of the defining representation $\mathbf{U}(1) \ni e^{i\alpha} \longmapsto (e^{i\alpha})^z \in \mathbf{U}(1)$

$$z \in \mathbb{Z} = \mathbf{weights}\,\mathbf{U}(1) \cong \mathbf{irrep}\,\mathbf{U}(1).$$

Any $\mathbf{U}(1)$-representation space is decomposable into irreducible complex 1-dimensional spaces (Fourier series). For a compact Lie algebra[1] the $\mathbf{U}(1)$-structure comes in self-dual representations via $\mathbf{SO}(2)$-Lie algebras that constitute the Cartan subalgebras of a compact Lie algebra. In the integer-power representations $\mathbf{SO}(2) \ni e^{i\sigma_3\alpha} \longmapsto (e^{i\sigma_3\alpha})^n \in \mathbf{SO}(2)$ the winding numbers come in pairs $\{\pm n\}$

$$n \in \mathbb{N}_0 \cong \mathbf{irrep}\,\mathbf{SO}(2).$$

The dimensions of representation spaces go with the rank r integer winding numbers as familiar from the spin group $\mathbf{SU}(2)$ (chapter "Spin, Rotations, and Position"). All representation spaces have a polynomial structure with r indeterminates (totally symmetric tensor powers) as familiar from the spherical polynomials and harmonics.

The simplest simple Lie operations of $\mathbf{SU}(2)$ and their representations are characteristic for all semisimple ones (chapter "Simple Lie Operations"). Compact Lie groups have an Euclidean structure, e.g., definite Killing forms. Therefore, it does not come as surprise that the regular Platonic polytopes play a prominent role in the weight and root diagrams of the simplest simple Lie algebras with rank 1, 2 and 3, e.g., for $\mathbf{SO}(3)$, $\mathbf{SO}(5)$ or $\mathbf{SU}(3)$ and $\mathbf{SU}(4)$, and, via the hierarchical buildup structure, for all semisimple Lie algebra representations.

After the general structures of representations of simple Lie algebras, they are listed explicitly for the four main series (A_r, C_r, B_r, D_r).

6.1 Simple Representations of Simple Lie Symmetries

All compact Lie algebra representations and hence all *finite dimensional* representations of a semisimple complex Lie algebra $L \in \underline{\mathbf{lag}}_{\mathbb{C}}$ are semisimple, i.e., decomposable into simple ones:

$$\mathcal{D} : L \longrightarrow \mathbf{AL}(V), \quad V = \bigoplus_i V_i, \quad \mathcal{D} = \bigoplus_i \mathcal{D}|_{V_i}.$$

A representation of a direct sum-product of two Lie algebras is equivalent to a representation on a tensor product of vector spaces:

$$\begin{aligned} &[L_1, L_2] = \{0\}, \quad \mathcal{D} : L_1 \oplus L_2 \longrightarrow \mathbf{AL}(V), \\ &V \cong V_1 \otimes V_2, \quad \mathcal{D} \cong \mathcal{D}^{V_1 \otimes V_2}, \quad \begin{cases} \mathcal{D}(l_1) \cong \mathcal{D}_1(l_1) \otimes \mathrm{id}_{V_2}, \\ \mathcal{D}(l_2) \cong \mathrm{id}_{V_1} \otimes \mathcal{D}_2(l_2). \end{cases} \end{aligned}$$

Therefore the representation structure of semisimple Lie algebras is determined by the simple representations of simple Lie algebras.

All finite dimensional (simple) representations of (semi)simple Lie algebras are characterized by weights with real or imaginary integer components,

[1] All Lie algebras considered in this chapter are assumed to be finite-dimensional.