7

QUANTUM ALGEBRAS

Quantum theory is a theory of really parametrizable operations acting on complex vector spaces with a conjugation. Quantum structures are unifying: The operations themselves are elements of their action spaces. Hence the classical distinction between operations and observables, e.g. between infinitesimal rotations and angular momenta, vanishes, in quantum theory the infinitesimal position rotations are identical with the angular momentum action.

Basically, the operational structure of quantum theory is algebraic, it does not start with states or with a Hilbert space or with "probability amplitudes," those concepts are needed for an ontology with particles and objects, necessary for a classical interpretation that arises by an experiment-induced projection of the operational algebraic structure. Hilbert spaces come with the definite unitarity of representations (chapter "Quantum Probability").

Linearity and concatenation of quantum operations are formalized with multilinearity (in distinction to "nonlinearity"). The multilinear structure for a vector space is its tensor algebra. The implementation of the basic vector space endomorphism Lie algebra in the form of inner algebra derivations leads to the *Fermi and Bose quantum algebra* of a vector space. Quantum elements are dual-product-induced equivalence classes in the tensor algebra, i.e., quantum algebras are tensor algebra quotient structures. The dual product of the quantum algebra underlying vector space leads to the *characteristic anticommutation and commutation relations of Fermi or Bose type* including the famous Born-Heisenberg position-momentum relation $[i\mathbf{p}, \mathbf{x}] = \hbar$.

The rich structure of quantum algebras, a mathematically basic and rather simple canonical factorization of the tensor algebra, will be considered in this chapter with respect to the implemented operations and the related invariants, with respect to their gradings and to the induced conjugations with the position-momentum formulation for the Bose quantum algebras.

The quantum algebras in this chapter are appropriate and enough for actions on finite-dimensional vector spaces, e.g., for irreducible representations of compact groups or of the time group $\mathbf{D}(1) \cong \mathbb{R}$. For faithful Hilbert representations of noncompact groups with infinite-dimensional spaces which occur in spacetime quantum theories, e.g., of the Poincaré group, they have to be generalized and play a role as value spaces (chapters "Massive Particle Quantum Fields" and "Harmonic Analysis").

The simplest examples are the Fermi and Bose oscillator quantum algebras for the abelian operation groups $\mathbf{U}(1)$ and $\mathbf{D}(1)$. They can be interpreted as the free algebras for two real 3-dimensional bracket algebras, for the Heisenberg Lie algebra $\log \mathbf{H}(1)$ with one position-momentum pair $[\mathbf{x}, \mathbf{p}] = \mathbf{I}$ in the Bose case and in the Fermi case for the hybrid Pauli Lie algebra $\log \mathbf{P}(1)$ with one creation-annihilation pair $\{\mathrm{u}^\star, \mathrm{u}\} = \mathbf{I}$.

The smallest simple compact group $\mathbf{SU}(2)$ gives rise to the spin quantum algebra of Fermi type which is used for spin $\frac{1}{2}$ particles, e.g., for electrons (chapter “Massive Particle Quantum Fields”). Adjoint Lie algebra representations with associated adjoint quantum algebras are exemplified by the position quantum algebra of Bose type as used in quantum mechanics with 3-dimensional position translations and the rotation group $\mathbf{SO}(3)$. In quantum field theories, adjoint quantum algebras are used for quantum gauge structures (chapter “Gauge Interactions”).

7.1 Quantization

Quantization connects *multilinearity with “canonical pairs.”* It is based on a pair of dual vector spaces (V, V^T), containing, e.g., position-momentum or creation-annihilation operators.

A quantum algebra is generated by the self-dual direct sum $\mathbf{V} = V \oplus V^T \cong \mathbb{K}^{2n}$ of a finite-dimensional vector space and its linear forms with the transposition sign $\epsilon = \pm 1$ (Fermi and Bose) defined by the symmetry property of the extended dual product

$$\mathbf{V} \times \mathbf{V} \longrightarrow \mathbb{K}, \begin{cases} \langle \mathbf{w}, \mathbf{v} \rangle = \langle u + \omega, v + \theta \rangle = \langle \omega, v \rangle + \langle u, \theta \rangle = \epsilon \langle \mathbf{v}, \mathbf{w} \rangle \\ \text{with } \langle \omega, v \rangle = \epsilon \langle v, \omega \rangle \text{ for } v \in V,\ \omega \in V^T. \end{cases}$$

All algebras for a vector space arise from its tensor algebra, e.g., Grassmann, polynomial, Clifford, or enveloping algebra. The tensor algebra $\bigotimes \mathbf{V}$ has two bilinear basic vector products: The ϵ-commutators $[\mathbf{w}, \mathbf{v}]_\epsilon \in \mathbf{V} \otimes \mathbf{V}$ which are power-2 tensors and the canonical products $\langle \mathbf{w}, \mathbf{v} \rangle \in \mathbb{K}$ which are scalars. In a *quantum algebra* $\mathbf{Q}_\epsilon(\mathbf{V})$ those two products are identified by working with equivalence classes with respect to the appropriate minimal, such an identity enforcing ideal $I(S_\epsilon^{\text{quant}})$:

$$\begin{aligned} S_\epsilon^{\text{quant}} &= \{\mathbf{w} \otimes \mathbf{v} + \epsilon \mathbf{v} \otimes \mathbf{w} - \langle \mathbf{w}, \mathbf{v} \rangle \bigm| \mathbf{v}, \mathbf{w} \in \mathbf{V}\} \subset \bigotimes \mathbf{V}, \\ \text{in } \mathbf{Q}_\epsilon(\mathbf{V}) &= \bigotimes \mathbf{V} / I(S_\epsilon^{\text{quant}}) : \begin{cases} [\mathbf{w}, \mathbf{v}]_\epsilon = \langle \mathbf{w}, \mathbf{v} \rangle, \\ [\omega, v]_\epsilon = \langle \omega, v \rangle, \\ [u, v]_\epsilon = 0, \quad [\omega, \theta]_\epsilon = 0. \end{cases} \end{aligned}$$

The embedding of a dual vector space pair into the tensor algebra with the basis independent factorization will be called *quantization*. For each vector space there exist both a *Fermi and a Bose quantum algebra* distinguished by the involutive transposition sign with $\epsilon^2 = 1$, i.e., by $\epsilon = \pm 1$ as *statistical sign*.

The main property of and motivation for such the tensor algebra factorization for a dual product-commutator identification is the implementation of