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QUANTUM PROBABILITY

Experiments measure numeric-valued properties of physical operations, especially invariants and eigenvalues like spin and its third direction or mass and energy-momenta. In quantum theory, the operations are formalized by endomorphisms of complex vector spaces with conjugation. The association of numbers to operators requires the study of the linear forms of an algebra, i.e., of the operation algebra dual, e.g., for a quantum algebra $\omega : \mathbf{Q}_\epsilon(\mathbb{K}^{2n}) \longrightarrow \mathbb{K}$. The endomorphisms of the quantum algebra underlying vector space $V \cong \mathbb{K}^n$ distinguish trace forms, which induce inner products, i.e., symmetric bilinear forms for real and sesquilinear forms for complex quantum algebras with a conjugation. The latter ones are the origin of the quantum characteristic "probability amplitudes" used in Hilbert spaces for the interpretation of experiments.

A quantum algebra with a conjugation determines "its" Hilbert spaces it is acting on. The unitary invariance group of an inner product of a complex basic vector space $V \cong \mathbb{C}^n$, definite or indefinite, and hence the associated conjugation can be determined by a representation of the real time group acting on it, e.g., given by a Hamiltonian in a complex quantum algebra $\mathbf{Q}_\epsilon(\mathbb{C}^{2n})$ with the conjugation implementing the time reflection. Abelian and nonabelian endomorphism algebras have two characteristic linear forms, induced by the trace and the "double trace," the abelian and nonabelian form. A quantum subalgebra with a positive inner product (prescalar product), sometimes, but not necessarily the full quantum algebra, is a pre-Hilbert space that in the associated Hilbert space allows a probability valuation of physical operations. Therefore, a complex representation of the really parametrized time transformations can be interpreted with the quantum characteristic "probability amplitude" structure, as established by Born.

In general, a real Lie group determines "its" Hilbert spaces (chapter "Harmonic Analysis"). The properties of Hilbert space vectors are given by their behavior with respect to the acting operations: physical objects are formalized by eigenvectors with respect to translations.

In addition to time translations, objects are acted on also with position translations. They lead to position orbits which, in nonrelativistic quantum theory, are described by Schrödinger's wave functions. They allow an interpretation of the quantum operations with a classical position space ontology and

give a position spread of the time translation eigenvectors and, for normalizable functions, position densities of probabilities.

First, general structures of algebra forms are considered - the transition from linear forms of an algebra to inner products, especially to scalar products for the related Hilbert spaces. Then concrete forms are looked for: Trace forms of endomorphism algebras have canonical extensions to quantum algebras. Finally, the familiar representations of the Heisenberg groups on the Hilbert spaces of square integrable position functions are given.

8.1 From Operator Algebra to Hilbert Spaces

Physical objects are formalized, in quantum theory, by Hilbert space vectors which are constructed as linear-form-induced equivalence classes of operations.

Some general properties and concepts for linear forms of vector spaces: A linear form $\omega : V \longrightarrow \mathbb{K}$ is injective on the classes $V/\operatorname{kern}\omega$ with the kernel. Forms with (conjugate) linear reflection can be conjugated too:

$$V \in *\underline{\mathbf{vec}}_{\mathbb{K}} \Rightarrow V^T \in *\underline{\mathbf{vec}}_{\mathbb{K}},\ \omega \overset{*}{\leftrightarrow} \omega^*;\ \text{with } \omega^*(v) = \overline{\omega(v^*)} \text{ for all } v \in V.$$

ω is *symmetric* (conjugation compatible) for $\omega = \omega^*$.

Two forms compose a *product form of the tensor product* inheriting properties of the factors:

$$\begin{array}{l} \omega_1 \otimes \omega_2 : V_1 \otimes V_2 \longrightarrow \mathbb{K},\quad (\omega_1 \otimes \omega_2)(v_1 \otimes v_2) = \omega_1(v_1)\omega_2(v_2), \\ V_1^T \otimes V_2^T \subseteq (V_1 \otimes V_2)^T, \text{ for finite dimensions } V_1^T \otimes V_2^T \cong (V_1 \otimes V_2)^T. \end{array}$$

A linear form is nontrivially *factorizable* (decomposable) if the vector space is nontrivially factorizable, $V = V_1 \otimes V_2$ with $V_{1,2} \neq \mathbb{K}$, with a corresponding form factorization $\omega = \omega_1 \otimes \omega_2$. Otherwise it is called a *nonfactorizable (irreducible)* form.

An inner product $\zeta : V \times V \longrightarrow \mathbb{K}$ of a vector space can be extended to its tensor products and the corresponding totally symmetric or antisymmetric subspaces, e.g., for power-2 tensors

$$\zeta^2 : \bigotimes^2 V \times \bigotimes^2 V \longrightarrow \mathbb{K},\quad \zeta^2(v_1 \otimes v_2, w_1 \otimes w_2) = \zeta(v_1, w_1)\zeta(v_2, w_2) \text{ etc.}$$

With a group G or Lie algebra L acting on a vector space, the action is given also on the forms by the corresponding dual representation. This defines G and L-invariant forms. $*$-symmetry is interpretable as time reflection invariance for a time-reflection-implementing conjugation.

The linear forms $A^T \in \underline{\mathbf{vec}}_{\mathbb{K}}$ of an associative algebra $A \in \underline{\mathbf{aag}}_{\mathbb{K}}$ (operator algebra), do not have to be compatible with the product structure, i.e., they do not have to be algebra morphisms for A in the abelian algebra $\mathbb{K}$ with the numbers, especially not in the case of the quantum characteristic nonabelian algebras. With respect to the *multiplication compatibility*, an algebra form can