9

SPECTRUM OF SPACETIME

In Wigner's classification, linear spacetime and free particles originate from one operational concept and its representations, from an affine subgroup with Lorentz transformations acting on translations. Why the free particles have the characteristic invariants, i.e., the observed masses m^2, spins J, and, for the additional internal $\mathbf{U}(1)$-operations, charge numbers z, is not explained by classifying the irreducible Hilbert representations of the Poincaré group. The actual spectrum of matter $(m^2, 2J, z) \in \mathbb{R}_+ \times \mathbb{N} \times \mathbb{Z}$ together with the normalization of particles and the coupling constants of interactions has to be understood by additional structures, e.g., by representations of a nonlinear spacetime model.

The multilinear algebra structure of quantum operations involves typical ensembles of representations ("towers of bound states"), which are products of one basic representation, defining the relevant operation group. Characteristic examples are the free states of translations, which are familiar from the equidistant linear spectrum of the harmonic oscillator; representations of time translations $\mathbb{R} \cong \mathbf{D}(1)$; and the bound states of the nonrelativistic hydrogen atom as representations of hyperbolic 3-dimensional position $\mathcal{Y}^3 \cong \mathbf{SO}_0(1,3)/\mathbf{SO}(3) \cong \mathbb{R}^3$ with the inverse squared energy spectrum.

A pointwise product of positive-type functions $d \in L^\infty(G)_+$ of a real Lie group is a positive-type function for the product representation,

$$d_1 \cdot d_2(g) = \langle a_1|D_1(g)|a_1\rangle\langle a_2|D_2(g)|a_2\rangle = \langle a_1, a_2|D_1 \otimes D_2(g)|a_1, a_2\rangle.$$

For the harmonic components, one has to use the convolution $\tilde{d}_1 * \tilde{d}_2$.

The characters (representation classes, dual group) as eigenvalues of the additive group $\check{\mathbb{R}}^d$—energies for time translations $\mathbb{R}$ and momenta for position translations $\mathbb{R}^3$—give rise to convolution algebras of the corresponding distributions (functions, measures). Nonlinear spacetime $\mathbf{D}(2) \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$ as a homogeneous space of the extended Lorentz group $\mathbf{GL}(\mathbb{C}^2)$ with tangent Minkowski translations $x \in \mathbb{R}^4$ can be represented by residues of Fourier transformed energy–momentum $q \in \check{\mathbb{R}}^4$ functions (chapter "Residual Spacetime Representations"). The representation-characterizing invariants arise as poles in the complex energy and momentum plane.

Product representations come with the product of representation coefficients, i.e., in a residual formulation with the convolution $*$ of (energy–)mo-

mentum distributions. The convolution itself picks up a residue,

$$* \sim \delta(q_1 + q_2 - q) \sim \mathrm{res}_{q_1+q_2=q}.$$

The convolution adds (energy–)momenta of singularity manifolds as imaginary and real eigenvalues for compact and noncompact representation invariants. The Radon (energy–)momentum measures are a convolution algebra, which reflects the pointwise multiplication property of the essentially bounded function classes:

$$\mathcal{M}(\check{\mathbb{R}}^n) * \mathcal{M}(\check{\mathbb{R}}^n) \subseteq \mathcal{M}(\check{\mathbb{R}}^n), \quad L^\infty(\mathbb{R}^n) \cdot L^\infty(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n).$$

In the Feynman integrals of special relativistic quantum field theory as convolutions of energy–momentum distributions, the on-shell parts for translation representations give product representation coefficients of the Poincaré group, i.e., energy–momentum distributions for free states (multiparticle measures, below). The off-shell interaction contributions are not convolutable. This is the origin of the "divergence" problem in quantum field theories with interactions. With respect to Poincaré group representations, the convolution of Feynman propagators makes no sense.

In this chapter the convolution structure of time, position, and spacetime representations is considered. In the end an attempt is made to determine, from eigenvalue equations, the spectrum of spacetime $\mathbf{D}(2) \cong \mathbb{R}^4_+ = \mathbf{D}(1) \times \mathcal{Y}^3$, i.e., invariant masses and normalizations of energy-momentum poles for the representations of the causal group, Lorentz compatibly embedded into nonlinear Minkowski spacetime. Perhaps one can characterize this as an attempt to find a Lorentz compatible solution of the bound state problem in a potential $V_3(r)$ with exponential and Yukawa contributions which has been given above (chapter "Residual Spacetime Representations") as the projection of the representation of nonlinear spacetime to representations of hyperbolic 3-position.

Only some illustrations of an explicit calculation are given for the determination of the particle properties as spectrum of a homogeneous spacetime model. If the proposal for the solution of such a difficult problem really goes in the right direction, both the qualitative foundations and the concrete realization require more work.

9.1 Convolutions for Abelian Groups

Product representations of translations $\mathbb{R}^n$ with sum and difference of the energy–momentum invariants arise as the pointwise product of positive-type functions $L^\infty(\mathbb{R}^n)_+$ or the convolution of positive energy-momentum Radon measures $\mathcal{M}(\check{\mathbb{R}}^n)_+$.

The simplest case is given for 1-dimensional translations, e.g., for time translations $t \in \mathbb{R}$ with an addition of the energy invariants in the irreducible