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LORENTZ OPERATIONS

Spacetime translations are characterized by a causality (order) compatible "metric" with indefinite $(1,3)$-signature, defining or defined by Lorentz transformations (chapter "Spacetime Translations"). In the complex formulation of quantum structures the noncompact Lorentz group also has to be represented in a unitary group - because of the unbounded group volume necessarily indefinite unitary for finite-dimensional nontrivial representations.

If the rotation group $\mathbf{SO}(3)$ for position translations $\mathbb{S} \cong \mathbb{R}^3$ with the spin Lie algebra[1] $A_1^c \cong (i\mathbb{R})^3$ and its Lie group $\mathbf{SU}(2)$ is represented by actions on complex vector spaces with canonical conjugation (chapter "Antistructures: The Real in the Complex"), it is embedded into representations of the doubled Lie algebra $A_1^c \hookrightarrow A_1^c \oplus iA_1^c \cong A_{(1,1)} \cong \mathbb{R}^6$. This involves an embedding for the Cartan subalgebras $i\mathbb{R} \hookrightarrow i\mathbb{R} \oplus \mathbb{R} = \mathbb{C}_\mathbb{R}$ and their groups $\mathbf{SO}(2) \hookrightarrow \mathbf{SO}(2) \times \mathbf{SO}_0(1,1) = \mathbf{SO}(\mathbb{C}^2_\mathbb{R})$. The subindex $\mathbb{R}$ in $\mathbb{C}_\mathbb{R}$ denotes a real structure represented in the complex, i.e., with a conjugation. For a less-cumbersome notation, it will be omitted in the following, only real Lie operations will be considered.

The doubled Lie algebra $A_{(1,1)}$ is the Lie algebra of the Lorentz group whose defining representation space gives a model for Minkowski spacetime $\mathbb{M} \cong \mathbb{R}^4$. The noncompact Lorentz structures arise by complexification of the compact spin structures $\mathbf{SO}(3) \hookrightarrow \mathbf{SO}(\mathbb{C}^3)$. The classes of the real 6-dimensional Lie group $\mathbf{SL}(\mathbb{C}^2) = \exp A_{(1,1)}$ with respect to its center constitute the orthochronous Lorentz group $\mathbf{SO}_0(1,3) \cong \mathbf{SL}(\mathbb{C}^2)/\mathbb{I}(2) \cong \mathbf{SO}(\mathbb{C}^3)$. The orientation manifold of spin groups in a Lorentz group is parametrizable by the real 3-dimensional noncompact symmetric boost space, the hyperboloid $\mathcal{Y}^3 \cong \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \cong \mathbf{SO}_0(1,3)/\mathbf{SO}(3)$. In such a complexification approach the causal order of Minkowski spacetime as $\mathbf{SL}(\mathbb{C}^2)$-representation space comes as a surprise. The connection between complexification and causality (order) is considered in more detail in the chapter "Spacetime as Unitary Operation Classes."

In this chapter all finite-dimensional irreducible $\mathbf{SL}(\mathbb{C}^2)$-representations are given. They arise by a doubling of the irreducible $\mathbf{SU}(2)$-representations starting from the Weyl doubling of the fundamental Pauli spinor representation. For those finite-dimensional representations, the integer winding numbers $\mathbb{Z}$

[1]In this chapter a Lie algebra structure of a vector space is defined up to linear equivalence.

as eigenvalues for compact spin $\mathbf{SU}(2)$ are paired with integers $\mathbb{Z}$ for the noncompact boosts $\mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2)$. The relevant unitary group for the complex finite-dimensional $\mathbf{SL}(\mathbb{C}^2)$-representations is the indefinite anticonjugation group $\mathbf{U}(2,2)$.

Definite unitary, i.e., Hilbert space representations of the group $\mathbf{SL}(\mathbb{C}^2)$ are, if faithful, necessarily infinite-dimensional; the noncompact boosts have eigenvalues from a continuous spectrum. They will be discussed in the chapter "Harmonic Analysis."

1.1 Spacetime Lie Algebras

1.1.1 Lorentz Lie Algebra

The operational structure for spacetime translations can be introduced as canonical complexification of the spin operations for position translations.

The Lie algebra $A_1^c \oplus iA_1^c \cong \mathbb{R}^6$, doubling the spin Lie algebra A_1^c, has as Lie brackets in a doubled orthogonal basis

$$\text{basis of } A_1^c \oplus iA_1^c: \quad \{l^a, b^a = il^a \,\big|\, a = 1,2,3\}, \quad \left\{ \begin{array}{ll} [l^a, l^b] & = -\epsilon^{abc} l^c, \\ [l^a, b^b] & = -\epsilon^{abc} b^c, \\ [b^a, b^b] & = +\epsilon^{abc} l^c. \end{array} \right.$$

The *Lorentz Lie algebra* $A_{(1,1)} \cong \mathbb{R}^6$ is the, up to linear equivalence, unique Lie algebra with the neutral signature $(3,3)$ for the Killing form. It allows Cartan decompositions $A_{(1,1)} \cong A_1^c \oplus iA_1^c$ into a compact 3-dimensional Lie subalgebra and a noncompact 3-dimensional vector subspace. It is simple with rank 2, i.e., its eigenvectors are characterized by two eigenvalues. From the A_1^c-Casimir element $-\frac{\delta_{ab}}{2} l^a \otimes l^b$, the inverse definite Killing form for the angular momenta, the complexification leads to two invariant power-2 tensors, the inverses of the two signature $(3,3)$ invariant forms for the Lorentz Lie algebra:

$$\begin{array}{ll} I_+(A_{(1,1)}) & = -\frac{\delta_{ab}}{4} (l^a \otimes l^b - b^a \otimes b^b), \\ I_-(A_{(1,1)}) & = -\frac{\delta_{ab}}{2} l^a \otimes b^b = -\frac{\delta_{ab}}{4} (l_+^a \otimes l_+^b - l_-^a \otimes l_-^b), \quad l_\pm^a = \frac{l^a \pm b^a}{\sqrt{2}}, \end{array}$$

where, $I_+(A_{(1,1)})$ as the inverse Killing form of $A_{(1,1)}$, is called the *Killing-Casimir element*, $I_-(A_{(1,1)})$ the *chiral Casimir element*. They generate all invariants for Lorentz transformations, i.e., the center of the enveloping algebra $\mathbf{E}(A_{(1,1)})$.

As A_1^c is isomorphic to the angular momentum Lie algebra $\log \mathbf{SO}(3)$ of the rotation group, so $A_{(1,1)}$ is isomorphic to $\log \mathbf{SO}(\mathbb{C}^3)$ or to the Lorentz Lie algebra $\log \mathbf{SO}_0(1,3)$ with angular momenta and boosts, in orthogonal bases

$$\begin{array}{lll} A_{(1,1)} & \cong \quad \log \mathbf{SO}(\mathbb{C}^3) & \cong \quad \log \mathbf{SO}_0(1,3), \\ \varphi_a l^a + \psi_a b^a & \cong \begin{pmatrix} 0 & -(\varphi_3 + i\psi_3) & \varphi_2 + i\psi_2 \\ \varphi_3 + i\psi_3 & 0 & -(\varphi_1 + i\psi_1) \\ -(\varphi_2 + i\psi_2) & \varphi_1 + i\psi_1 & 0 \end{pmatrix} & \cong \left(\begin{array}{c|ccc} 0 & \psi_1 & \psi_2 & \psi_3 \\ \hline \psi_1 & 0 & -\varphi_3 & \varphi_2 \\ \psi_2 & \varphi_3 & 0 & -\varphi_1 \\ \psi_3 & -\varphi_2 & \varphi_1 & 0 \end{array} \right), \end{array}$$

$$\begin{array}{ll} l^a & \cong \frac{\epsilon^{abc}}{2} \mathbf{l}^{bc}, \\ b^a & \cong \mathbf{l}^{a0} = -\mathbf{l}^{0a}, \\ a, b & = 1,2,3, \end{array} \quad \left\{ \begin{array}{ll} [l^a, l^b] & \cong [\frac{\epsilon^{aef}}{2} \mathbf{l}^{ef}, \frac{\epsilon^{bcd}}{2} \mathbf{l}^{cd}] = \mathbf{l}^{ba}, \\ [b^a, b^b] & \cong [\mathbf{l}^{a0}, \mathbf{l}^{b0}] = \mathbf{l}^{ab}, \\ [b^a, l^b] & \cong [\mathbf{l}^{a0}, \frac{\epsilon^{bcd}}{2} \mathbf{l}^{cd}] = \epsilon^{bcd} \frac{\eta^{ad} \mathbf{l}^{c0} - \eta^{ac} \mathbf{l}^{d0}}{2} = -\epsilon^{abc} \mathbf{l}^{c0}. \end{array} \right.$$