2

SPACETIME AS UNITARY OPERATION CLASSES

In quantum theory, time and position are really parametrized operations acting on complex vector spaces. The causal homogeneous manifolds that will be discussed in this chapter are n^2-dimensional generalizations of 1-dimensional time and 4-dimensional spacetime. They are constituted by classes of compact unitary transformations in complex linear ones as suggested by the Cartan presentation of Minkowski spacetime by Hermitian (2×2) matrices (chapter "Lorentz Operations"). The description of these causal manifolds with real rank n clarifies the structures of 4-dimensional spacetime with real rank 2 as the physically most important case.

From a mathematical point of view, the first sections of this chapter contain, in physical terms, a reformulation of familiar structures of the stellar algebras (C*-algebras) $\mathbf{AL}(\mathbb{C}^n)$ with $n \times n$ matrices acting on $\mathbb{C}^n$-isomorphic Hilbert spaces (chapter "Quantum Probability"). If spacetime translations constitute a real vector subspace in complex linear transformations $x : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $x = x^\star$, they are, from the outset, recognizable as binary relations in the sense of Leibniz, for Minkowski spacetime $n = 2$ as binary spinor relations.

The polar decomposition of the full linear group $\mathbf{GL}(\mathbb{C}^n)$ into the group $\mathbf{U}(n)$ with the phases and the symmetric space $\mathbf{D}(n)$ with the absolute values – the uniquely defined positive cone in the stellar algebra of all complex $(n \times n)$ matrices – is proposed to establish the dichotomy of compact internal ("chargelike") and noncompact external ("spacetimelike") degrees of freedom respectively as used in quantum field theories for $n = 2$ leading to the compact hyperisospin group $\mathbf{U}(2)$ (chapter "Gauge Interactions") and the noncompact nonlinear spacetime $\mathbf{D}(2)$ with tangent Minkowski translations $\mathbb{R}^4$.

2.1 Spacetime Translations

Cartan's parametrization of the spacetime translations (real 4-dimensional Minkowski vector space) uses the Hermitian complex 2×2 matrices (chapter

"Lorentz Operations"). Together with the time translations (real numbers)

$$x = \begin{cases} t = \overline{t} & \in \mathbb{C}_{\mathbb{R}}(1) = i\mathbb{R} \oplus \mathbb{R}, & n = 1, \\ \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = x^\star & \in \mathbb{C}_{\mathbb{R}}(2) = (i\mathbb{R})^4 \oplus \mathbb{R}^4, & n = 2, \end{cases}$$

they should be used as illustrations for general n.

The complex $n \times n$ matrices $z \in \mathbb{C}(n) = \mathbf{AL}(\mathbb{C}^n)$ constitute, with the $\mathbf{U}(n)$-conjugation $\star$, a *stellar algebra* (C*-algebra) as endomorphisms of a $\mathbb{C}^n$-isomorphic Hilbert space. They are decomposable into two isomorphic vector spaces of real dimension n^2:

$$z = \tfrac{i}{2}\gamma + x \in \mathbb{C}_{\mathbb{R}}(n) = i\mathbb{R}(n) \oplus \mathbb{R}(n) \cong \mathbb{R}^{2n^2}.$$

The vector subspace $\mathbb{R}(n)$ is called the *matrix parametrization of the spacetime translations* with $n \in \mathbb{N}$ the *real rank of spacetime*.

A basis for $\mathbb{C}_{\mathbb{R}}(n)$ is given by generalized Weyl matrices

$$z = z_j \sigma(n)^j \cong z_A^{\dot{A}}, \quad A, \dot{A} = 1, \ldots, n \text{ with } \{\sigma(n)^j\}_{j=0}^{n^2-1} = \{\mathbf{1}_n, \sigma(n)^a\}_{a=1}^{n^2-1}$$

where $\mathbf{1}_n$ is the unit matrix and $\sigma(n)^a$ for $n \geq 2$ are $(n^2 - 1)$ generalized Hermitian traceless Pauli matrices, i.e., three Pauli matrices $\vec{\sigma}$ for $n = 2$, eight Gell-Mann matrices $\sigma(3)^a = \lambda^a$ for $n = 3$, etc. (chapter "Simple Lie Operations").

The determinant defines the *abelian projection* on the complex numbers ($\star$ monoid morphism):

$$\det : \mathbb{C}_{\mathbb{R}}(n) \longrightarrow \mathbb{C}_{\mathbb{R}}, \quad \det z^\star = \overline{\det z}.$$

By polarization, i.e., by an appropriate combination of $(z_1 \pm z_2 \pm \cdots \pm z_n)^n$, one obtains a totally symmetric $\star$-compatible multilinear form, generalizing the well-known bilinear form of the Minkowski translations $\mathbb{R}(2)$:

$$\begin{aligned} \eta :\; & \mathbb{C}_{\mathbb{R}}(n) \times \cdots \times \mathbb{C}_{\mathbb{R}}(n) \longrightarrow \mathbb{C}, \\ & (z_1, \ldots, z_n) \longmapsto \eta(z_1, \ldots, z_n) = \epsilon^{A_1 \ldots A_n} \epsilon_{\dot{A}_1 \ldots \dot{A}_n} (z_1)_{A_1}^{\dot{A}_1} \cdots (z_n)_{A_n}^{\dot{A}_n}, \\ n = 1 :\; & \eta(z) = \det z = z, \\ n = 2 :\; & \eta(z_1, z_2) = \tfrac{(z_1+z_2)^2 - (z_1 - z_2)^2}{4}, \quad \operatorname{sign} \eta|_{\mathbb{R}(2)} = (1, 3). \end{aligned}$$

The trace and the traceless parts of a translation are called a time translation and a position translation respectively. The position translation $\vec{x}$ denotes a traceless $(n \times n)$ matrix and, in this connection, $x_0 \cong x_0 \mathbf{1}_n$:

$$\begin{aligned} \operatorname{tr} \mathbb{C}_{\mathbb{R}}(n) &= i \operatorname{tr} \mathbb{R}(n) \oplus \operatorname{tr} \mathbb{R}(n), \quad \operatorname{tr} \mathbb{R}(n) \cong \mathbb{R}, \\ \mathbb{C}_{\mathbb{R}}(n)_0 &= \{z \in \mathbb{C}_{\mathbb{R}}(n) \bigm| \operatorname{tr} z = 0\} = i\mathbb{R}(n)_0 \oplus \mathbb{R}(n)_0, \quad \mathbb{R}(n)_0 \cong \mathbb{R}^{n^2-1}, \\ x &= x_j \sigma(n)^j = x_0 \mathbf{1}_n + x_a \sigma(n)^a = x_0 + \vec{x}. \end{aligned}$$

A spacetime decomposition into time and position translation subspaces is incompatible with the determinant, since in general, $\det(x + y) \neq \det x + \det y$ for $n \geq 2$.