3

PROPAGATORS

Feynman propagators characterize the spacetime behavior of particles. They will be introduced as Lorentz compatible relativistic distributions of matrix elements of time representations. The particle interpretation is discussed in the chapters "Massive Particle Quantum Fields" and "Massless Quantum Fields."

Representations of the causal group $\mathbf{D}(1) \cong \exp \mathbb{R}$, generated by and isomorphic to the time translations $\mathbb{R}$, can be embedded, by position distribution, into a Lorentz-action-compatible framework. The invariant time operation eigenvalues (energies, frequencies) are distributed by energy-momentum $(q_0, \vec{q})$-measures (generalized functions) supported by the Lorentz invariant mass hyperboloid $q^2 = m^2$. As special relativistic supplement for the compact time representation matrix elements $e^{\pm i|q_0|t} \in \mathbf{U}(1) \cong \mathbf{SO}(2)$, there arise $r = 0$-regular spherical waves $\frac{\sin|\vec{q}|r}{r}$, $|\vec{x}| = r$, which are representation coefficients of the Euclidean group $\mathbf{SO}(3) \vec{\times} \mathbb{R}^3$ (chapter "The Kepler Factor"). The causal time representations $e^{\pm i|q_0 t|}$ are supplemented by $r = 0$-singular Yukawa potentials $\frac{e^{-|\vec{Q}|r}}{r}$.

The relation of relativistic distributions of time representations to representations of the Poincaré group $\mathbf{SO}_0(1,3) \vec{\times} \mathbb{R}^4$ is discussed in the chapters "Harmonic Analysis" and "Residual Spacetime Representations."

3.1 Point Measures for Energies

To prepare the relativistic embedding, the time representation matrix elements are formulated as Fourier transformed energy measures. The continuous eigenvalues of the irreducible unitary time representations can be embedded as the real axis $m \in \mathbb{R}$ into the complex energy plane. The Dirac distributions of the energies define *point supported measures of the complex energy plane*. They can also be written as a loop integration around an energy pole:

$$1 = \int dE\ \delta(m - E) = \oint \frac{dE}{2i\pi} \frac{1}{E - m} \text{ for } m \in \mathbb{R}.$$

Here the following notation with Lebesgue measure dE is used:

$\int dE$ for $\int_{-\infty}^{\infty} dE = \int_{\mathbb{R}} dE$ on the real axis,
$\oint dE$ for a positive (counterclockwise) loop around all poles.

All distributions (generalized functions) used for propagators are tempered $\mathcal{S}'(\mathbb{R}^d)$ with the Fourier isomorphism $\mathcal{S}'(\check{\mathbb{R}}^d) \cong \mathcal{S}'(\mathbb{R}^d)$.

The Dirac point measure, equivalent to a *residue* $\int dE\ \delta(E-m)f(E) = \oint \frac{dE}{2i\pi}\frac{f(E)}{E-m}$, is the real part of a complex generalized function where the principal value function, denoted by the subscript P, comes as imaginary part:

$$a \in \mathbb{R},\quad \pm\frac{1}{i\pi}\frac{1}{a\mp io} = \delta(a) \pm \frac{1}{i\pi}\frac{1}{a_{\mathrm{P}}} \iff \left\{\begin{array}{lll} \delta(a) & = \frac{1}{2i\pi}\left[\frac{1}{a-io} - \frac{1}{a+io}\right] & = \frac{1}{\pi}\frac{o}{a^2+o^2}, \\ \frac{1}{a_{\mathrm{P}}} & = \frac{1}{2}\left[\frac{1}{a-io} + \frac{1}{a+io}\right] & = \frac{a}{a^2+o^2}. \end{array}\right.$$

The symbol o in the generalized function prescribes a pole with a *real positive* $o > 0$, an integration on the real axis and, afterward, the limit $o \to 0$.

The complex point measures with a pole in the energy plane are Fourier transforms of the *advanced and retarded* time representations

$$\vartheta(\pm t)e^{imt} = \pm\int \frac{dE}{2i\pi}\frac{1}{E\mp io-m}e^{iEt}.$$

The distributional imaginary part determines the time direction, the upper half-plane pole for $E - io$ leads to support by the future, the lower half-plane pole $E + io$ to support by the past.

With those measures and functions time representation matrix elements can be written in different forms, with a closed loop integration, with a Dirac measure, or with a time-ordered principal value integration:

$$\begin{array}{ll} \mathbb{R} \longrightarrow \mathbf{U}(1) \ni e^{imt} & = \oint \frac{dE}{2i\pi}\frac{1}{E-m}e^{iEt} = \int dE\ \delta(m-E)e^{iEt} \\ & = \epsilon(t)\int \frac{dE}{i\pi}\frac{1}{E_{\mathrm{P}}-m}e^{iEt}. \end{array}$$

The self-dual time representations with the trigonometric functions use an energy measure self-dually supported by $\pm m$:

$$\begin{array}{ll} \mathbb{R} \longrightarrow \mathbf{SO}(2) & \ni \begin{pmatrix} \cos mt & i\sin mt \\ i\sin mt & \cos mt \end{pmatrix}, \\ \text{with } \begin{pmatrix} \cos mt \\ i\sin mt \end{pmatrix} & = \int dE\ \epsilon(m)\begin{pmatrix} m \\ E \end{pmatrix}\delta(m^2-E^2)e^{iEt} \\ & = \int dE\ \epsilon(E)\begin{pmatrix} E \\ m \end{pmatrix}\delta(m^2-E^2)e^{iEt} \\ & = \oint \frac{dE}{i\pi}\frac{1}{E^2-m^2}\begin{pmatrix} E \\ m \end{pmatrix}e^{iEt} = \epsilon(t)\int \frac{dE}{i\pi}\frac{1}{E_{\mathrm{P}}^2-m^2}\begin{pmatrix} E \\ m \end{pmatrix}e^{iEt}. \end{array}$$

The causal time representations have as energy measures

$$\begin{pmatrix} 1 \\ \epsilon(t) \end{pmatrix}e^{\pm i|mt|} = \begin{pmatrix} 1 \\ \epsilon(t) \end{pmatrix}(\cos mt \pm \epsilon(mt)i\sin mt) = \int \frac{dE}{i\pi}\begin{pmatrix} \pm|m| \\ E \end{pmatrix}\frac{1}{E^2\mp io-m^2}e^{iEt}.$$

Representations with finite closed integration contours in the complex plane like e^{imt} for the group $\mathbb{R}$ obey homogeneous differential $\frac{d}{dt}$ equations, those with infinite unclosed contours like $\sin|mt|$ for $\mathbb{R} = \mathbb{R}_+ \uplus \mathbb{R}_-$ as ordered double cone inhomogeneous ones.