7

HARMONIC ANALYSIS

In addition to physical properties that are characterized by a *rational number*, such as an integer electromagnetic charge number or a (half)integer spin (chapter "Rational Quantum Numbers"), particles have properties that seem[1] to be taken from a *continuous spectrum*, especially their masses and the gauge coupling constants, e.g., the fine structure constant normalizing the electromagnetic $\mathbf{U}(1)$-Lie algebra in the spacetime-formulated interaction.

All quantum numbers (invariants and eigenvalues) for compact Lie groups, including finite groups, are rational and, as well as the finite-dimensionality of their irreducible representation spaces, ultimately related to integer winding numbers as powers $e^{i\alpha} \longmapsto (e^{i\alpha})^Z$ of the circle group (torus) $\mathbf{U}(1) = \exp i\mathbb{R}$ (chapter "Rational Quantum Numbers"). Lie operations for continuous quantum numbers have to come from a *noncompact Lie group*, as familiar from the eigenvalues for the representations of the causal group $\mathbf{D}(1) = \exp \mathbb{R} \ni e^t \longmapsto (e^t)^{im} \in \mathbf{U}(1)$, e.g., the energies $m \in \mathbb{R}$ for scattering waves. The eigenvalues for irreducible representations are characterized in the following table displaying the twofold dichotomy compact-noncompact and abelian-nonabelian:

	compact	noncompact
abelian	$\mathbf{U}(1) \longrightarrow \mathbf{U}(1)$ $e^{i\alpha} \longmapsto e^{Zi\alpha}$ $Z \in \mathbb{Z}$	$\left\{ \begin{array}{l} \mathbf{D}(1) \longrightarrow \mathbf{U}(1) \\ \mathbf{D}(1) \longrightarrow \mathbf{SU}(1,1) \end{array} \right.$ $e^{\beta} \longmapsto e^{(im+\gamma)\beta}$ $im + \gamma \in i\mathbb{R} + \mathbb{R}$
nonabelian $n \geq 2$	$\mathbf{SU}(2) \longrightarrow \mathbf{SU}(1+2J)$ $u \longmapsto \bigvee^{2J} u, \; 2J = \lvert Z \rvert$ $Z \in \mathbb{Z}$	$\left\{ \begin{array}{l} \mathbf{SL}(\mathbb{C}^2) \longrightarrow \text{compact} \\ \mathbf{SL}(\mathbb{C}^2) \longrightarrow \text{noncompact} \end{array} \right.$ $\mathbb{Z} \times [i\mathbb{R} + \mathbb{R}]$

irreducible group representations with example weights

Throughout this chapter complex vector spaces $V \in \underline{\mathbf{vec}}_{\mathbb{C}}$ for representations of locally compact groups, especially of real finite-dimensional Lie groups $G \in \underline{\mathbf{lgrp}}_{\mathbb{R}}$ with positive and G-invariant *Haar measure*, an indispensible important tool for continuous groups, are considered. The complex numbers are

[1]Since experimental numbers come with errors, one can never be sure whether they are from a rational or a continuous spectrum. Numerologists develop great skills for integer graduations of experimental numbers by a few units, e.g., for the fine structure constant $\alpha \sim 2^{-2}\pi^{-3}$. With a few exceptions, e.g., Balmer's formula, numerologists find dead-end quantitative coincidences without qualitative insights.

used with the canonical conjugation, i.e., as the doubled reals. Complex represented real Lie group operations have to come as unitary automorphisms, i.e., with definite or indefinite conjugations. Definite unitary representations are also called *Hilbert representations*. Definite and indefinite unitary groups have a compact and noncompact parameter space respectively. Therefore, spaces with faithful Hilbert representations of noncompact groups have to be infinite-dimensional.

$\mathbf{rep}\, G$ denotes the equivalence classes of G-representations with respect to the intertwining isomorphisms, the classes of the irreducible (Hilbert) representations are called the *(definite) group dual* (also dual group space):

$$\mathbf{irrep}\, G = \check{G}, \quad \mathbf{irrep}_+ G = \check{G}_+.$$

The irreducible representations are characterized by the invariants of the group and its Lie algebra.

To give a first survey: Finite groups, e.g., cyclic or permutation groups, with the discrete topology are special cases of compact groups, e.g., unitary groups, which are special cases of locally compact groups, e.g., complex linear groups

$$\begin{array}{rcccccc} \text{group } G: & \text{finite} & \subset & \text{compact} & \subset & \text{locally compact} \\ \text{e.g., abelian} & \mathbb{I}(n) & \subset & \mathbf{U}(1) & \subset & \mathbf{GL}(\mathbb{C}) \\ \text{special} & \mathbf{G}(n)_+ & \subset & \mathbf{SU}(n) & \subset & \mathbf{SL}(\mathbb{C}^n) \\ \text{general} & \mathbf{G}(n) & \subset & \mathbf{U}(n) & \subset & \mathbf{GL}(\mathbb{C}^n) \\ \text{group dual } \check{G}_+: & \text{finite} & \subset & \text{countable} & \subset & \text{continuous} \end{array}$$

Harmonic analysis is the classification of the irreducible representations of a group and the decomposition of the group mappings, e.g., physical fields as spacetime mappings, taken as a "huge" representation vector space, into irreducible group representation spaces. Especially for noncompact groups, it connects intimately algebraic with topological and measure structures. It will be based on the rather straightforward stuctures for finite groups. Finite-dimensional Hilbert representations build up the representation structure of finite and compact groups. This is briefly recapitulated in the first part of this chapter. In contrast to the completely understood and explicitly known representations of compact and abelian groups with a general theory, the analogous situation is much more difficult and complicated for noncompact nonabelian groups with their individual pecularities. In the following, only an orientation - sometimes very sloppy - is given without doing justice to the many topological and measure-related structures, which should be looked at in more detail in the mathematical literature.

The harmonic analysis of functions on a locally compact group involves, as dual partner for a Haar measure $d^G k$ of the group, a positive and G-invariant *Plancherel measure* of the group dual with the characterizing invariants

$$\text{Haar measure } d^G k \leftrightarrow d^{\check{G}_+} D \text{ Plancherel measure.}$$

This is exemplified for the rotation group $\mathbf{SO}(3)$ with Euler angle parametrized normalized Haar measure $\int_{\mathbf{SO}(3)} \frac{d\chi d\varphi d\cos\theta}{(4\pi)^2}$ by the associated Plancherel measure