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RESIDUAL SPACETIME REPRESENTATIONS

In Feynman propagators (chapter "Propagators") with energy-momentum poles $\frac{i}{\pi}\frac{1}{q^2+io-m^2}$, the Fourier transform of the real part, i.e., of the distribution $\delta(q^2-m^2)$, supported by the energy-momentum hyperboloid ("on-shell"), represents free particles with real momenta $\vec{q}^2 = q_0^2 - m^2 > 0$ by coefficients of the translations in the Poincaré group, e.g., $e^{iq_0 t}\frac{\sin|\vec{q}|r}{r}$. The imaginary "momenta" $q_0^2 - m^2 = -Q^2 < 0$ in the principal value "off-shell" distribution $\frac{1}{q_{\rm P}^2-m^2}$ leads to interactions, e.g., to Yukawa interactions in $e^{iq_0 t}\frac{e^{-|Q|r}}{r}$. Spacetime interactions are supported by the causal bicone. The harmonic analysis of the future cone $\mathbf{D}(2) \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$ (unitary relativity) as a nonlinear homogeneous spacetime model, i.e., of the mappings $W^{\mathbf{D}(2)}$ of the full linear group, constant on $\mathbf{U}(2)$, into $\mathbf{U}(2)$-representation spaces $W \cong \mathbb{C}^{1+2J}$, involves the representations of the acting extended Lorentz group $\mathbf{GL}(\mathbb{C}^2)$, which have to be used for spacetime interactions. Free particle fields are not complete for the harmonic analysis of nonlinear spacetime, genuine interaction fields are necessary [4, 14]. Interactions cannot be expanded completely with free particles.

Representations of linear and nonlinear spacetime embed time and position representations. Representation coefficients of 3-dimensional hyperbolic position $\mathcal{Y}^3$ as symmetric space for Lorentz operations $\mathbf{SO}_0(1,3)$ can be written with Fourier transformed 3-sphere momentum measures (chapter "The Kepler Factor") as seen in Hilbert-space-valued Schrödinger-bound state functions, e.g., for the hydrogen ground state $e^{-|m|r} = \int \frac{d^3q}{2\pi^2}\frac{2|m|}{(\vec{q}^2+m^2)^2}e^{-i\vec{q}\vec{x}}$. These representations of nonlinear position $\mathcal{Y}^3$ with a dipole singularity sphere for imaginary momenta $\vec{q}^2 = -m^2$ have to be embedded into causally supported representation coefficients of nonlinear spacetime $\mathbf{D}(2) \cong \mathbf{D}(1) \times \mathcal{Y}^3$. The embedding energy-momentum distributions do not describe free particles: The Lorentz invariant mass for the representation of the position degree of freedom comes as a singularity in a higher-order pole, starting with a dipole distribution $\frac{d^4q}{(q^2-m^2)^2}$, as required by Lorentz compatible embedding of the 3-sphere measures $\frac{d^3q}{}\frac{2|Q|}{(\vec{q}^2+Q^2)^2}$ with energy-dependent invariant $Q^2 = m^2 - q_0^2$.

Multipole energy-momentum distributions lead, via their Fourier transforms with appropriate integration contours, to residual representations of

symmetric spaces. Representations of time (harmonic oscillator), of position (scattering and bound waves), of spacetime translations (on-shell part of Feynman propagators), and of nonlinear spacetime (spheres, hyperboloids, multipole interactions) will be formulated in the language of residual representations with their characterizing invariant singularities.

8.1 Linear and Nonlinear Spacetime

Minkowski translations $\mathbb{R}^4$ with position translations $\mathbb{R}^3$ contain as substructure Cartan translations $\mathbb{R}^2$ with time and 1-dimensional position translations $\mathbb{R}$ (chapter "Spacetime as Unitary Operation Classes"). Cartan and Minkowski translations are parametrizable by Hermitian (2×2) matrices, diagonal for Cartan spacetime,

$$\begin{tikzcd} t \in \mathbb{R} \arrow[r] & x^0 + \sigma_3 x^3 \in \mathbb{R}^2 \arrow[r] & x = x^0 + \vec{x} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \in \mathbb{R}^4. \\ x^3 \in \mathbb{R} \arrow[r] \arrow[ur] & \vec{x} \in \mathbb{R}^3 \arrow[ur] & \end{tikzcd}$$

A group $\mathbf{D}(1) = \exp \mathbb{R}$ acts both on the time and the 1-dimensional position translations. For Cartan translations, it is rearranged from $\mathbf{D}(1) \times \mathbf{D}(1)$ to $\mathbf{D}(\mathbf{1}_2) \times \mathbf{SO}_0(1,1)$ with the rotation free orthochronous Lorentz group (self-dual dilations). Together with the position rotations $\mathbf{SO}(3)$, it is embedded into the $\mathbf{D}(\mathbf{1}_4)$-extended Poincaré group,

$$\begin{tikzcd} \mathbf{D}(1) \vec{\times} \mathbb{R} \arrow[r] & {[\mathbf{D}(1) \times \mathbf{SO}_0(1,1)] \vec{\times} \mathbb{R}^2} \arrow[r] & {[\mathbf{D}(1) \times \mathbf{SO}_0(1,3)] \vec{\times} \mathbb{R}^4.} \\ \mathbf{D}(1) \vec{\times} \mathbb{R} \arrow[r] \arrow[ur] & {[\mathbf{D}(1) \times \mathbf{SO}(3)] \vec{\times} \mathbb{R}^3} \arrow[ur] & \end{tikzcd}$$

Spacetime has an order structure: Time future is embedded into Cartan and Minkowski future,

$$\begin{array}{rcll} \mathbb{R}_+ \ni t = \vartheta(t)t & \hookrightarrow & \vartheta(x^2)\vartheta(x^0)(x^0 + \sigma_3 x^3) & = x \in \mathbb{R}^2_+, \\ & \hookrightarrow & \vartheta(x^2)\vartheta(x^0)(x^0 + \vec{x}) & = x \in \mathbb{R}^4_+. \end{array}$$

The futures are used as noncompact spaces (open cones without "skin"), i.e., without the strict presence $x = 0$ and without lightlike translations for nontrivial position $s = 1, 3$,

$$x \in \mathbb{R}^{1+s}_+ \Rightarrow x^2 > 0, \; s = 0, 1, 3.$$

Time future is the causal group $\mathbf{D}(1) = \exp \mathbb{R}$,

$$\mathbb{R}_+ \ni t \;\; = e^{\psi_0} \in \mathbf{D}(1) \cong \mathbf{GL}(\mathbb{C})/\mathbf{U}(1).$$

Cartan future is the direct product of causal group and abelian Lorentz group,

$$x = e^{\psi_0 + \sigma_3 \psi} \in \mathbb{R}^2_+ \cong \mathbf{D}(1) \times \mathbf{SO}_0(1,1).$$