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Small Ball Probabilities and Semi-metrics

13.1 Introduction

All the theoretical advances in nonparametric statistics for functional variables presented above show powerfully the key role played by the small ball probability function, both on the several different hypothesess made throughout this book and on the rates of convergence. Clearly, this function depends on the topological structure existing on the functional semi-metric space and which is induced by the semi-metric itself.

The main aim of this chapter is to describe precisely all the theoretical links existing between the small ball probability functions and the semi-metrics. In particular, we will present some examples of usual processes for which the small ball probability function can be evaluated explicitely. Purely functional examples are presented in Section 13.3, while Sections 13.4 and 13.5 will go back to standard finite dimensional ones. All these theoretical considerations will complete the empirical ideas discussed in Chapter 3 as well as the different case studies presented throughout the book which indicated that functional nonparametric methods could have quite interesting effect on real data situations if (and only if) one has selected a suitable semi-metric. Finally, we will see that in functional nonparametric statistics the semi-metric modelling turns to be the key point both for practical and theoretical issues.

As a by-product, we will see that even if infinite dimensional setting is the main purpose of this book, the general approach that we have followed here can be of interest in finite dimensional nonparametric problems. We will see in Section 13.4 how this approach allows us to extend several results existing in usual one-dimensional nonparametric statistical problems. We will also see in Section 13.5 how the approach can provide a new way to attack the curse of dimensionality in multivariate nonparametric problems.

13.2 The Role of Small Ball Probabilities

Recall that $\boldsymbol{\mathcal{X}}$ is a random variable taking values into some metric-space (E, d), and that χ is a fixed (deterministic) element of E. For any of the various nonparametric problems treated earlier, each asymptotic result is directly linked with the measure (with respect to the probability distribution of $\boldsymbol{\mathcal{X}}$) of a ball of center χ. It turns that the following function:

$$\varphi_\chi(.) = P(\boldsymbol{\mathcal{X}} \in B(\chi, .)),$$

plays a crucial role. More precisely, the key point is the behaviour of the function $\varphi(.)$ when the radius of the ball tends to zero, and this is the reason why it is called *small ball probability function* or equivalently *concentration function*.

To fix the ideas, look for instance at the result provided in Theorem 6.11 for kernel functional regression estimation (but keep in mind that everything said here will concern equivalently all other rates of convergence given earlier in this book). Theorem 6.11 stated that, under suitable conditions, the kernel nonparametric estimate $\widehat{r}$ constructed with a bandwidth h was converging to the true nonlinear regression operator r with a rate of convergence of the form:

$$O\left(h^\beta\right) + O\left(\sqrt{\frac{\log n}{n\varphi_\chi(h)}}\right).$$

While the first component comes from the bias of the estimate and depends only on the smoothness of the operator r, the second one comes from the variablity of the estimate and is therefore highly linked with the concentration of the data. The Lipschitz parameter β as defined in condition (5.12) is linked with the smoothness of r, while the small ball probability function $\varphi_\chi(h)$ directly measures the concentration of the functional variable $\boldsymbol{\mathcal{X}}$. The less dispersed are the functional data $\boldsymbol{\mathcal{X}}_i, \dots \boldsymbol{\mathcal{X}}_n$, the more efficient will be the estimator. With other words, the more concentrated the random variable $\boldsymbol{\mathcal{X}}$, the higher will be the small ball probability function φ_χ and the faster will be the rate of convergence of the functional nonparametric estimate to the true target operator.

At this stage, it is natural to wish the small ball probability function to be as high as possible to avoid possible overdispersion effects. This probabilistic point of view can be however balanced by some considerations of the topological structure of the functional space. Indeed, the notion of concentration (and therefore the function φ_χ itself) is directly linked with the structure of the space E, in such a way that what could appear to be a purely probabilistic question turns to be primarily a topological one. Therefore, one could expect to be able to reduce the possible overdispersion effects just by changing the topological structure on the space E, that is by changing the semi-metric d.

Finally, it should be emphasized that the roles of the probability distribution of the functional variable $\boldsymbol{\mathcal{X}}$ and of the semi-metric d are completely