

Some Selected Asymptotics

This chapter states a few asymptotic results, all related to the problem of predicting some real random response variable given some explanatory variable which is allowed to be of infinite dimension. We have arbitrarily decided to organize the presentation around the problems of estimating the three functional predictors defined in Chapter 5: the regression, the conditional median and the conditional mode. Additional results on conditional quantiles, on conditional functional c.d.f. and on conditional density estimation problems will also be given. Even if this field of statistics is quite new, it was impossible to present all the results which are actually available in the literature. The necessary selection was done according to two wishes: providing self-contained and detailed proofs of some key results without hiding the main features of functional problems with too much technicality. This is the reason why some of the results are not presented under the most sophisticated sets of assumptions. A final section will complete these results by discussing the relevant bibliography and by giving a prominent place to the statements of open problems.

6.1 Introduction

The aim of this chapter is to present some asymptotic results linked with nonparametric estimation of the three functional predictors already defined in Section 5.2: conditional expectation, conditional median and conditional mode. The nonparametric predictors to be used are based on kernel smoothing ideas and are those defined before in Section 5.4. All the results are presented in terms of almost complete convergence. This stochastic mode of convergence may appear quite unusual for some people, but it has been selected because it has two important advantages. First it is stronger than almost sure convergence and convergence in probability, and second it is easier to prove than the almost sure consistency itself. To help the reader who is not very familiar with this kind of asymptotics, Appendix A recalls the basic definitions coming

with almost complete convergence and shows its links with other usual modes of convergence.

The asymptotic results are divided into two parts: convergence results are first stated in Section 6.2 and the rates of convergence are stated precisely later on Section 6.3. Of course, the treatment of each of the three functional predictors (conditional expectation, conditional median and conditional mode) involves necessarily the study of its associated nonlinear operator: the regression, the conditional c.d.f. and the conditional density function. Therefore, each of these sections is divided in five parts: one for each of the three functional nonparametric predictors, one for the extension of conditional median results to conditional quantiles, and a last one for summarizing and completing different results concerning the estimation of the nonlinear operators associated with the conditional distribution of the process. As it is basically the case in un-functional (i.e., finite dimensional) situations, the statement of consistency results relies on continuity assumptions on the predictor or on the nonlinear operator to be estimated, and naturally the results of Section 6.2 will be stated on continuity-type models like those described in Section 5.3. As in finite dimensional settings, the specification of the rates of convergence relies on additional smoothness conditions and the results of Section 6.3 will be stated under Lipschitz-type conditions under the functional parts of the nonlinear operators to be considered. Rather than looking for the most sophisticated technicalities, we decided to emphasize on results for which the specificities of the infinite dimensionality of the explanatory variables can be highlighted. Each of the results presented is accompanied by a complete proof.

A final Section 6.4 is devoted to the discussion of all the results given in this chapter. This discussion is organized around three ideas. First, we present an up-to-date survey of the quite few bibliography existing in this new setting of nonparametric statistics for functional prediction problems. Second, to emphasize the specificity of the functional feature of the problem, we will show how the results behave in finite dimensional setting. All the results of this chapter lead us to think that large parts of the nonparametric knowledges for finite dimensional statistics could be transplanted to infinite dimensional settings (depending, of course, on suitable functional adaptations). Therefore, to share this hope with the statisticians community and to encourage further investigation in this direction, the third and last (but not least) thread of these comments will be to release several theoretical open problems.

6.2 Almost Complete Convergence

6.2.1 Regression Estimation

We focus on the pointwise almost complete convergence of the functional kernel estimator of the regression $r(\mathcal{X}) = \mathbb{E}(Y|\mathcal{X})$, which was defined in 5.23. Note that we can write