We return to interpolating sequences and their Blaschke products, and in particular to the surprising part they play in unraveling the maximal ideal space of $H^\infty$. In this chapter three topics are discussed.

1. Analytic structure in $\mathcal{M}\setminus D$, and its relation to interpolating sequences. This theory, due to Kenneth Hoffman, occupies Sections 1 and 2. The theory rests on two factorization theorems for Blaschke products.

2. Two generalizations of the theorem that a harmonic interpolating sequence is an $H^\infty$ interpolating sequence. One of these generalizations is the theorem that a sequence is an interpolating sequence if its closure in $\mathcal{M}$ is homeomorphic to the Stone–Čech compactification of the integers. The key to these generalizations is a real-variables argument determining when one Poisson kernel can be approximated by convex combinations of other Poisson kernels. This idea is developed in Sections 3 and 4.

3. A more recent theorem, due to Peter Jones, that refines the Douglas–Rudin theorem. Here the analysis is not done on the upper half plane, but on the boundary.

The three topics have little interdependence, and they can be studied separately.

1. Analytic Discs in $\mathcal{M}$

It will be convenient to think of two copies of the unit disc. First there is $D = \{z : |z| < 1\}$, the natural domain of $H^\infty$ functions and an open dense subset of $\mathcal{M} = \mathcal{M}_{H^\infty}$. The second disc $\mathcal{D} = \{\zeta : |\zeta| < 1\}$ will be the coordinate space for many abstract discs in $\mathcal{M}$, including $D$ itself. Points of $\mathcal{D}$ will always
be denoted $\zeta$. When $z \in D$, the mapping $L_z : D \to D$ defined by

$$L_z(\zeta) = \frac{\zeta + z}{1 + \overline{z}\zeta}$$

cordinatizes $D$ so that $z$ becomes the origin.

A continuous mapping $F : D \to M$ is \textit{analytic} if $f \circ F$ is analytic on $D$ whenever $f \in H^\infty$. An \textit{analytic disc} in $M$ is a one-to-one analytic map $L : D \to M$. With an analytic disc we do not distinguish between the map $L$ and its image $L(D)$. It is not required that $L$ be a homeomorphism, and there are natural examples where it cannot be one (see Exercise 8). The mappings $L_z$ above are examples of analytic discs. In this and the next section, we describe all the analytic maps into $M$ and we present Hoffman’s fascinating theory connecting analytic discs to interpolating sequences. But first we need some general facts about possible analytic structure in $M$.

The \textit{pseudohyperbolic distance} between $m_1 \in M$ and $m_2 \in M$ is

$$\rho(m_1, m_2) = \sup\{|f(m_2)| : f \in H^\infty, \|f\|_\infty \leq 1, f(m_1) = 0\}.$$ 

On $D$ this definition of $\rho(m_1, m_2)$ coincides with the earlier one for $\rho(z_1, z_2)$ introduced in Chapter I, because if $m_j = z_j \in D$, then by Schwarz’s lemma

$$\rho(m_1, m_2) = \left|\frac{z_1 - z_2}{1 - \overline{z}_2z_1}\right|.$$ 

Moreover, the distance $\rho(m_1, m_2)$ on $M$ retains many of the properties of $\rho(z_1, z_2).$ If $f \in H^\infty, \|f\|_\infty \leq 1$, then

$$\rho(f(m_1), f(m_2)) \leq \rho(m_1, m_2).$$

because $g(z) = (f(z) - f(m_1))/(1 - f(m_1)f(z))$ satisfies $\|g\|_\infty \leq 1, g(m_1) = 0,$ and $|g(m_2)| = \rho(f(m_1), f(m_2)).$ Choosing \{f_n\} such that $\|f_n\|_\infty \leq 1, f_n(m_1) = 0,$ and $|f_n(m_2)| \to \rho(m_1, m_2),$ we see that

$$\rho(m_1, m_2) = \sup\{\rho(f(m_1), f(m_2)) : f \in H^\infty, \|f\| \leq 1\}.$$ 

By Lemma 1.4 of Chapter I, we have

$$\frac{(1.1)\rho(m_0, m_2) - \rho(m_2, m_1)}{1 - \rho(m_0, m_2)\rho(m_2, m_1)} \leq \rho(m_0, m_1) \leq \frac{\rho(m_0, m_2) + \rho(m_2, m_1)}{1 + \rho(m_0, m_2)\rho(m_2, m_1)}$$

$m_j \in M, j = 0, 1, 2.$ Indeed, the left-hand inequality follows from that lemma by taking $\rho(f(m_0), f(m_2))$ close to $\rho(m_0, m_2)$ and noting that $(s - t)/(1 - st)$ is decreasing in $t$ when $0 \leq s, t \leq 1;$ while the right-hand inequality follows by taking $\rho(f(m_0), f(m_1))$ close to $\rho(m_0, m_1)$ and noting that $(s + t)/(1 + st)$ increases in both $s$ and $t$ when $0 \leq s, t \leq 1.$

Clearly $\rho(m_1, m_2) \leq 1,$ and by (1.1) the relation

$$m_1 \sim m_2 \text{ iff } \rho(m_1, m_2) < 1$$