Chapter 8

RELIABILITY AND CREDIT RISK MODELS

In this chapter, the reader will first find a short summary of the basic notions of reliability and then the semi-Markov extensions. After that, the classical problem of credit risk is also presented together with an analogy with reliability and it is shown how semi-Markov models are useful for this important topic of finance in connection with the new rules of the Basel Committee.

1 CLASSICAL RELIABILITY THEORY

Reliability theory is mainly concerned with the security of material fittings. A first distinction must be made between simple and complex structures. For a simple structure, it is possible to define what is called the lifetime of the considered system, defined as the r.v. $T$ representing the time interval between time 0 and the time of the first failure, failure meaning that the system is out. A complex system is composed of several simple components, from which failures have an impact, more or less important, on the way the system is working.

1.1 Basic Concepts

Let us consider a simple structure called the reliability system $S$ having r.v. $T$ as lifetime, $T$ being defined on the probability space $(\Omega, \mathcal{F}, P)$.

**Definition 1.1** The reliability function of $S$ is given by the function $U$ defined as

$$U(t) = P(T > t), t \in [0, \infty).$$

(1.1)

$U(t)$ represents the probability that no failure happens before $t$. If $F$ represents the distribution function of $T$, it is clear that for all non-negative $t$:

$$U(t) = 1 - F(t).$$

(1.2)

If the density function $f$ of $T$ exists, we obtain:

$$U(t) = \int_0^\infty f(u)du.$$  

(1.3)

From now on, we always assume that the $f$ or the derivative of $U$ exists.

**Definition 1.2** The function $r$, defined as
\[ r(t) = \frac{f(t)}{1 - F(t)} = \frac{-U'(t)}{U(t)}, \quad t > 0, \quad (1.4) \]

is called the failure rate of the component.

Its meaning is simple: let us consider a time \( t \) such that the event \( \{T > t\} \) occurs. From basic definitions of conditional probability (relation (6.4) of Chapter 1) and from relation (1.2), we can successively write:

\[
P(T \leq t + dt | T > t) = \frac{P(t < T \leq t + dt)}{P(T > t)}
\]

\[
= \frac{f(t)dt}{U(t)}
\]

\[
= - \frac{U'(t)}{U(t)} dt
\]

\[
= r(t)dt.
\]

Consequently, \( r(t)dt \) simply represents the conditional probability of having a failure in the infinitesimal time interval \((t, t+dt)\) given that the component has no failure before or at time \( t \). So, the value of the failure rate at time \( t \) is a risk measure to have suddenly a failure just after time \( t \).

By integration, relation (1.4) gives:

\[
U(t) = e^{-\int_r(t)du} \quad (1.6)
\]

provided that we suppose that \( U(0)=1 \).

From the last relation, it is clear that any non-negative function can be a failure rate if the following two conditions are satisfied:

- the function \( r \) is integrable on the positive half real line,

- \( \int_0^\infty r(u)du = \infty \). \quad (1.7)

The mean lifetime \( \overline{T} \) is just the mean for the d.f. \( F \).

By integration by parts, it is possible to show that

\[
\overline{T} = \int_0^\infty U(t)dt \quad (1.8)
\]

and similarly if the variance \( \sigma^2 \) exists:

\[
\sigma^2 = 2\int_0^\infty tU(t)dt \quad (1.9)
\]

### 1.2 Classification Of Failure Rates

The first classification of failure rate types was given by Barlow and Proschan (1965) with the following definition.