Eigenwavelets of the Wave Equation

Gerald Kaiser

Signals and Waves
3803 Tonkawa Trail #2
Austin, TX 78756
USA
www.wavelets.com
kaiser@wavelets.com

To Carlos Berenstein on his 60th birthday.

Summary. We study a class of localized solutions of the wave equation, called eigenwavelets, obtained by extending its fundamental solutions to complex spacetime in the sense of hyperfunctions. The imaginary spacetime variables $y$, which form a timelike vector, act as scale parameters generalizing the scale variable of wavelets in one dimension. They determine the shape of the wavelets in spacetime, making them pulsed beams that can be focused as tightly as desired around a single ray by letting $y$ approach the light cone. Furthermore, the absence of any sidelobes makes them especially attractive for communications, remote sensing and other applications using acoustic waves. (A similar set of “electromagnetic eigenwavelets” exists for Maxwell’s equations.) I review the basic ideas in Minkowski space $\mathbb{R}^{3,1}$, then compute sources whose realization should make it possible to radiate and absorb such wavelets. This motivates an extension of Huygens’ principle allowing equivalent sources to be represented on shells instead of surfaces surrounding a bounded source.

1 Extension of wave functions to complex spacetime

The ideas to be presented here affirm that complex analysis resonates deeply in “real” physical and geometric settings, and so they are close in spirit to the work of Carlos Berenstein (see [BG91, BG95, B98], for example), to whom this volume is dedicated.

Acoustic and electromagnetic wavelets were first constructed in [K94]. It was shown that solutions of homogeneous (i.e., sourceless) scalar and vector wave equations in Minkowski space $\mathbb{R}^{3,1}$ extend naturally to complex spacetime, and the wavelets were defined as the Riesz duals of evaluation maps acting on spaces of such holomorphic solutions. The sourceless wavelets then split naturally into retarded and advanced parts emitted and absorbed, respectively, by sources located on branch cuts needed to make these parts single valued. Later work [K3, K4] was aimed at the construction of realizable source distributions which, when synthesized, would act as antennas radiating and receiving the wavelets. Two difficulties with this approach
have been (a) that the computed sources are quite singular, consisting of multiple surface layers that may be difficult to realize in practice, and (b) in the electromagnetic case the sources appeared to require a nonvanishing magnetic charge distribution, which cannot be realized as no magnetic monopoles have been observed in nature. In this paper, we resolve the first difficulty by replacing the spheroidal surface supporting the sources in [K3, K4] by a spheroidal shell. It is shown in [K4a] that the second difficulty can be overcome using Hertz potentials, which give a charge-current distribution due solely to bound electric charges confined to the shell.

Although our constructions generalize to other dimensions, we shall concentrate here on the physical case of the Minkowski space $\mathbb{R}^{3,1}$. Let

$$x = (r, t), \quad y = (a, b) \in \mathbb{R}^{3,1}$$

be real spacetime vectors and define the complex causal tube

$$T = \{x - iy \in \mathbb{C}^4 : y \text{ is timelike, i.e., } |b| > |a|\}.$$  \hspace{1cm} (2)

It was shown in [K94, K3] that solutions of the homogeneous wave equation

$$\Box f_0(x) \equiv (\partial_t^2 - \Delta) f_0(r, t) = 0$$

extend naturally to analytic functions $\tilde{f}_0(x - iy)$ in $T$ in the sense that

$$\lim_{y \to +0} \{\tilde{f}_0(x - iy) - \tilde{f}_0(x + iy)\} = f_0(x),$$

where $y \to +0$ means that $y$ approaches the origin within the future cone, i.e., with $b > |a|$. This kind of extension to complex domains is familiar in hyperfunction theory; see [K88, KS99], for example. We now show that even when the wave function has a source, i.e.,

$$\Box f(x) = 4\pi g(x),$$

it extends analytically to $T$ outside a spacetime region determined by the source. It will suffice to do this for the retarded propagator

$$G(x) = \frac{\delta(t - r)}{r},$$

which is the unique causal fundamental solution:

$$\Box G(x) = 4\pi \delta(t)\delta(r) = 4\pi \delta(x), \quad G(r, t) = 0 \quad \forall t < 0.$$  \hspace{1cm} (7)

If the source $g$ is supported in a compact spacetime region $W$, the unique causal solution of (5) is given by

$$f(x) = \int_W dx' G(x - x')g(x').$$

Assume for the moment that $G(x)$ has been extended to $\tilde{G}(x - iy)$. Then we define the source of $\tilde{G}$ as the distribution $\delta$ in real spacetime given by