BOREL QUANTIZATION
AND NONLINEAR QUANTUM MECHANICS

A Review of Developments
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1. Introduction

In the first Symmetries in Science meeting in 1979 at Carbondale we presented preliminary results for a quantization of the classical kinematic for non–relativistic systems which are localized and moving on a smooth manifold $M$. Our paper [7] ‘On Global Properties of Quantum Systems’ was published in the Proceedings of Symmetries in Science series. We developed subsequently (with Bernd Angermann) [8, 9] a quantization method on smooth manifolds — the ‘Quantum Borel Kinematics’ (QBK); for a recent review see [10]. In 1992 a suitable time dependence was proposed (with Jerry Goldin) (see the review [11] and a more general ‘Borel Quantization’ (BQ) which emerged from geometrical and topological considerations; it indicated a nonlinear extension of quantum mechanics. We participated in some of the later editions of Symmetries in Science series, often together with members of the ‘Clausthal group’, e.g. Vlado Dobrev, Jerry Goldin, Wieland Groth, Jörg Hennig, Wolfgang Lücke, Hans-Jürgen Mann, Peter Nattermann, Wolfgang Scherer, Christoph Schulte, Pavel Štovíček and Reidun Twarock. The results, different aspects and applications of Borel Quantization can be found the volumes of ‘Symmetries in Science’.
Our interest in quantum mechanics on manifolds was connected with the following situation: During 1970–1980 some of our colleagues in quantum theory and in particle physics thought that Lie groups and their representations are a major key to model and to understand particle physics. In this context we worked e.g. on spectrum generating algebras and on embeddings of physical Lie algebras. Based on Mackey’s theory of induced representations we wrote a paper [12] on a quantization of particles moving on homogeneous \( G \)-spaces. We realized that the geometry of the \( G \)-space does not contain ‘enough’ information for a time evolution on \( G \). Furthermore, we failed to generalize Mackey’s method to physical important non–homogeneous spaces. Group theory was obviously a very successful model, but it was too ‘rigid’: If one chooses the group and its representation, the complete mathematical framework is already given; there does not appear the flexibility which one wants for a description of physical systems. Hence those mathematical formalisms which are ‘close’ to group theory and which are in addition more ‘flexible’ became interesting. Among such formalisms are: nonlinear and non–integrable representations of Lie algebras and their deformations in the sense of Gerstenhaber. A further promising field for a geometric modelling are differential geometrical and algebraic notions on \( M \). Here one views physical laws e.g. as relation between geometrical or algebraic objects living on \( M \). Following the pioneering papers of George Mackey [13] and Irving Segal [14], we found a path to understand quantizations of a system on a topologically nontrivial configuration space and how the quantized system ‘feels’ the topology. This leads to Quantum Borel Kinematics characterized by topological quantum numbers and one additional quantum number \( D \). This \( D \) is connected with the structure of the infinite–dimensional Lie algebra spanned by quantized kinematical operators.

In our kinematical design a physical interpretation of \( D \) was obscure. It became more transparent in 1988 when Jerry Goldin and one of the authors (HDD) realized that two approaches are equivalent: the quantized Borel kinematics on Euclidean configuration spaces and the representations of non–relativistic current algebras on multiparticle configuration spaces for indistinguishable objects found by Goldin and co-workers [15]. The quantum number \( D \) appears in front of an additional term in the generalized momentum operator as well as in the momentum current. Jerry Goldin and HDD introduced, based on this observation, a generic time dependence for pure states and derived a family of nonlinear Schrödinger equations [16] — DG equations — with nonlinear term proportional to \( D \). Special generalizations to mixed states (von Neumann equations) are known [17]. A direct connection of the DG family to certain nonlinear gauge transformations [18, 19] and an interpretation of \( D \) through nonlinear transformation [20] was elaborated.

Some of these developments are reviewed and commented in this contribution.