CHAPTER 7

VIBRATIONS OF ISOTROPIC BEAMS AND PLATES

7.1 Introduction

Through the previous chapter, the static behavior of beams, rods, columns and plates has been treated to determine displacements, stresses, and buckling loads. This is important because many structures are stiffness critical (maximum deflections are limited) or strength critical (maximum stresses are limited). In Chapter 6, the elastic stability of these structures was treated because that is a third way in which structures can be rendered useless. In most cases when a structure becomes elastically unstable, it cannot fulfill its structural purpose.

In this chapter, the vibration of beams and plates is studied in some detail. Many textbooks have been written dealing with this subject, but here, only an introduction is made to show how one approaches and deals with such problems.

In linear vibrations, both natural vibration and forced vibrations are important. The former deals with natural characteristic of any elastic body, and these natural vibrations occur at discrete frequencies, depending on the geometry and material systems only. Such problems (like buckling) are eigenvalue problems, the natural frequencies are the eigenvalues, and the displacement field associated with each natural frequency are the eigenfunctions. One remembers that in a simple spring-mass system, there is one natural frequency and mode shape; in a system of two springs and two masses, there are two natural frequencies and two mode shapes. In a continuous elastic system, theoretically there are an infinite number of natural frequencies, and a mode shape associated with each.

Forced vibrations occur when an elastic body is subjected to a time dependent force or forces. In that case the response to the forced vibrations can be viewed as a linear superposition of all the eigenfunctions (vibrations modes), each with an amplitude determined by the form of the forcing function. In forced vibrations, the forces can be cyclic (harmonic vibration) or non-cyclic, including shock loads (those which occur over very small times).
7.2 Natural Vibrations of Beams

Consider again the beam flexure equation discussed previously.

\[
\frac{d^4w}{dx^4} = q(x). \tag{7.1}
\]

It is seen that the forcing function \(q(x)\) is written in terms of force per unit length. Using d’Alembert’s Principle for vibration, an inertial term can be written which is the mass times the acceleration per unit length. Also the forcing function can be a function of time, and of course the lateral deflections will be a function of both spatial and temporal coordinates. The result is that (7.1) becomes, for the flexural vibration,

\[
\frac{d^4w}{dx^4} = q(x, t) - \rho_m A \frac{\partial^2w}{\partial t^2}. \tag{7.2}
\]

In the above \(\rho_m\) is the mass density of the beam material, and \(A\) is the beam cross-sectional area, both of which are taken here as constants for simplicity.

As stated previously, natural vibrations are functions of the beam material properties and geometry only, and are inherent properties of the elastic body – independent of any load. Thus, for natural vibrations, \(q(x, t)\) is set equal to zero, and (7.2) becomes

\[
\frac{d^4w}{dx^4} + \rho_m A \frac{\partial^2w}{\partial t^2} = 0. \tag{7.3}
\]

To solve this equation to obtain \(w(x, t)\), in general, one can assume \(w(x, t) = X(x)T(t)\), a separable solution, use separation of variables to obtain a spatial function \(X(x)\) which satisfies all of the boundary conditions, and an harmonic function for \(T(t)\), and thus arrive at a characteristic set of variables to satisfy (7.3) and its boundary conditions. In that process the natural frequencies and mode shapes are determined.

By way of a specific example, consider the beam to be simply supported at each end. Then the spatial function is a sine function such that

\[
w(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \sin \omega_n t \tag{7.4}
\]

where \(A_n\) is the amplitude, and \(\omega_n\) is the natural circular frequency in radians per unit time for the \(n\)th vibrational mode.

Substituting (7.4) into (7.3) results in:

\[
\sum_{n=1}^{\infty} A_n \left[ \frac{n^4 \pi^4}{L^4} - \omega_n^2 \rho_m A \right] \sin \frac{n\pi x}{L} \sin \omega_n t = 0. \tag{7.5}
\]